SOLUTIONS

AP Calculus Problems
Part II AB and BC

1986 – 2000

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Martin Rudolph

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An important notice from
Nicholas J. Restivo, Executive Director of

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Solution pages are identified by year, type of exam (AB or BC), and problem number.

Identification of each page appears both at the top and bottom of the page. The following conventions are observed:

- Years are listed in order from 1986 through to 2000.
- AB exams are listed before BC exams.
- Problems and solutions for each exam are listed in sequential order from problem 1.
- Problems and solutions common to both the AB and BC examinations are listed under the AB designation.

FOREWORD

Periodically, the College Board's Advanced Placement Program provides approved, complete solutions for the Part II problems on AP Examinations in Calculus. College Board Online offers the latest AP Program information to teachers on the web. The address is: www.collegeboard.org/ap/html/indx001.html. In addition, over 14 publications about Calculus are offered by the program. For an AP Publications Order Form write to: Advanced Placement Program, P.O. Box 6670, Princeton, NJ 08541-6670. To help teachers of Advanced Placement Calculus and their students, the Mathematical Olympiads for Elementary and Middle Schools (MOEMS), a not-for-profit public foundation, sponsors the production of this Solutions Book. The books for each of the fourteen years from 1982 through 1995 were written by the team Judith Broadwin (Jericho High School, NY) and Dr. George Lenchner (Executive Director, MOEMS). Upon Dr. Lenchner's retirement in 1996, Martin Rudolph (Oceanside High School, NY) was added to the team.

The following people served as consultants:

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Mr. Peter Hollenstein, Malverne High School, NY (now retired);
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The manuscript was prepared for publication by Nancy Paugh, Ramona Himpler, and Martin Rudolph. Ramona is responsible for the first set of computer solutions and Martin for the remaining sets of computer solutions.

The book is distributed by MOEMS and requests for information should be sent to:

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USING THIS BOOK

Some teachers ask all students to purchase a copy of the Solutions Book for AP Calculus, while others prefer to have a copy just for themselves.

We asked an outstanding teacher of AP Calculus to tell us how she uses the Solutions Book. Her response follows.

> *As a teacher, I use the Solutions Book (which students purchase) as another text. The AP problems are so rich and varied that I refer to them as illustrative problems all year. If students use the book correctly (and most do), they try the problem themselves and then check their solution with the book. Sometimes the student's approach may be different from that of the book. In such cases, we discuss the merits of each. This results in thinking of different ways of solving problems and the relative efficiency of each.*
>
> *According to former students, the Solutions Book and the notes they write in their copy serve as a valuable reference when they go on to college.*

On the other hand, a teacher may feel that when some students are assigned AP problems, they will copy from the Solutions Book. Another teacher may argue that if Solutions Books are not available, those students would copy from other students.

We suggest that teachers explain to students the procedures and attitudes that lead to intellectual growth and development when using a resource like the Solutions Book. It is important for students to hear this throughout their school career. When a student attempts a problem without help, the book's solution should be hidden. Upon completion, the student should compare his or her solution with that of the book. If the student's solution is different from the book's or has special features, the teacher should discuss it with the class.

If a student cannot do a problem, the student should look at the answer or look at the diagram for clues. Another strategy is to reveal one line of the book's solution at a time until the student is able to continue without help. In this case, the student should be advised to retry the problem one or two weeks later to see if the strategy has been retained.

Used this way, the Solutions Book will become an important resource for the student's growth and mathematical development.

INDEX BY TOPICS

Some topics which appear below may no longer be required. For changes in the AP Calculus courses, consult the latest *Advanced Placement Course Description, Mathematics: Calculus AB, Calculus BC.* Each of the topics listed below gives the AP Calculus examinations of the most recent fifteen-year period which are related to the topic.

Coding: (1) **87BC1ab** denotes the 1987 BC exam, problem 1, parts a & b; (2) **93AB4(BC3)** indicates that problem 4 of the 1993 AB exam and problem 3 of the BC exam are identical.

Integration Methods: 87BC3ab(parts) 90BC2(parts)

Interval / Convergence (see Series): 87BC4b 88BC4 91BC5c 94BC5b 96BC2b 00BC3b

Lagrange Error Bound: 99BC4b

Maximum, Minimum: 86AB1c 86AB5ab 87AB4a 87BC2c 88AB4b($2xe^x$) 88BC1c 89BC3 90AB5c 90BC3b 91AB5a 91BC1a 92AB3cd 93AB1c 93AB4(BC3)a($\ln x$) 94AB1b 94AB4c($\ln x$) 94BC4 95AB6d 96BC1b 96AB(BC)4c 97AB4a 97AB(BC)5b 98AB(BC)2 99AB4d 99AB(BC)5c 99BC1b 00BC4cd

Mean Value and Rolles' Theorems: 89AB1c 93AB3(BC1)a 97AB2c 99AB(BC)3b

Motion in a Plane (see Vectors)

Motion on a Line: 86AB3($\ln x$) 87AB1 88AB2 89AB3 90AB(BC)1 91BC1 92AB2a 93AB2(e^x) 94AB4($\ln x$) 95AB2 97AB1 98AB3ac 99AB1 00AB(BC)2

Parametric Equations: 87BC5(velocity) 89BC4 92BC3(e^x) 95BC1 96BC6b 97BC1 98BC6 99BC1

Polar Coordinates: 90BC4(area) 93BC4

Precalculus: 86AB1a(zeros) 86AB2ab(symmetry, asymptotes) 87AB2a(D(f)) 88AB1ab(D(f), symmetry) 89AB1a 89AB4abc(D(f), asymptotes) 90AB2ac 91AB3c 91AB4(absolute value) 92AB3ab

Rectangle Approximation Method (RAM): 98AB3d 99AB(BC)3a

Related Rates: 87AB5 88BC3 90AB4 91AB6 92AB6 92BC5ab 94AB5(BC2) 95AB5(BC3) 96AB5c(BC5a) 99AB6cd

Riemann Sums: 98AB3d 99AB(BC)3a

Second Derivative Test: 99AB4d 99BC4c

Series: 86BC5(Taylor) 87BC4ac(Taylor) 88BC4 90BC5(Taylor) 91BC5(Taylor) 92BC6 93BC5(Taylor) 94BC5(Taylor) 95BC4(Taylor) 96BC2 97BC2(Taylor) 98BC3(Taylor) 99BC4(Taylor Polynomial) 00BC3(Taylor Polynomial)

Slope Fields and Euler's Method: 98BC4ab 99BC6b

Tangents, Normals: 86AB1b 87AB2d 87BC2b 88AB1d 89AB1b 89BC4b(parametric equations) 90BC3a 91AB3ab 92AB1c 92AB4(BC1)b 92AB5(BC2)a(e^x) 92BC3(e^x) 93AB3(BC1)a 94AB1a 95AB3bc 96AB(BC)4bc 96AB6 97AB2b 97AB5c 97BC4b 98AB4ab 98AB6bc 99AB4ac 99AB6ab 99BC6a

Total Change Given a Rate of Change: 96AB(BC)3d 98AB(BC)5d

Trapezoidal Approximation: 94AB6a 96AB(BC)3c

Vectors: 87BC5 92BC3(e^x) 93BC2 95BC1 96BC6bc 97BC1 98BC6 99BC1 00BC4

Volume of Solid of Revolution: 86AB6b 87AB3ab 87BC3bc($\ln x$) 88AB3ab 88BC2b 89AB2bc 89BC2b 90AB3bc 90BC2ab 91AB2bc 91BC3b 92AB5(BC2)c(e^x) 93AB3(BC1)c 94AB2(BC1)bc(e^x) 95AB4(BC2)c 96AB5a 96BC1a 97AB2d 98AB1cd 98BC1bc 99AB(BC)2bc 00AB(BC)1b

Volume by Slicing: 88BC5 91BC3c 96AB2c 97BC3c 00AB(BC)1c

Work: 92BC5c 96BC5b

1986 – AB1

1. Let f be the function defined by $f(x) = 7 - 15x + 9x^2 - x^3$ for all real numbers x.
   (a) Find the zeros of f.
   (b) Write an equation of the line tangent to the graph of f at $x = 2$.
   (c) Find the x-coordinates of all points of inflection of f. Justify your answer.

---

a.) $7 - 15x + 9x^2 - x^3 = 0$

$f(1) = 0$

$$\begin{array}{rrrr|l} -1 & 9 & -15 & 7 & \underline{1} \\ & -1 & 8 & -7 & \\ \hline -1 & 8 & -7 & 0 & \end{array}$$

$-x^2 + 8x - 7 = 0 \Rightarrow (-x + 7)(x - 1) = 0 \Rightarrow x = 7, 1$

$\boxed{\text{The zeros of } f \text{ are } 1, 1, 7}$

b.) $f'(x) = -3x^2 + 18x - 15$
$f'(2) = -12 + 36 - 15 = 9$

$f(2) = 7 - 30 + 36 - 8 = 5$

$$\boxed{\begin{array}{c} y - 5 = 9(x - 2) \\ \underline{\text{OR}} \\ y = 9x - 13 \end{array}}$$

c.) $f''(x) = -6x + 18 = 0$ when $x = 3$

$$f''(x) \quad \underset{}{+} \quad \underset{3}{|} \quad - \quad \longrightarrow x$$

f is concave upward for $x < 3$

f is concave downward for $x > 3$

$\therefore$ $\boxed{f \text{ has an inflection point when } x = 3.}$

1986 - AB2

2. Let f be the function given by $f(x) = \dfrac{9x^2 - 36}{x^2 - 9}$.

(a) Describe the symmetry of the graph of f.

(b) Write an equation for each vertical and each horizontal asymptote of f.

(c) Find the intervals on which f is increasing.

(d) Using the results found in parts (a), (b), and (c), sketch the graph of f on the axes below.

---

a.) $f(-x) = \dfrac{9(-x)^2 - 36}{(-x)^2 - 9} = \dfrac{9x^2 - 36}{x^2 - 9} = f(x)$

$\therefore$ y-axis symmetry

b.) Vertical asymptotes: $x = 3$, $x = -3$

Horizontal asymptote: $y = 9$ since $\lim\limits_{x \to \infty} \dfrac{9x^2 - 36}{x^2 - 9} = 9$

c.) $f'(x) = \dfrac{(x^2-9)(18x) - (9x^2-36)(2x)}{(x^2-9)^2} = \dfrac{18x^3 - 162x - 18x^3 + 72x}{(x^2-9)^2}$

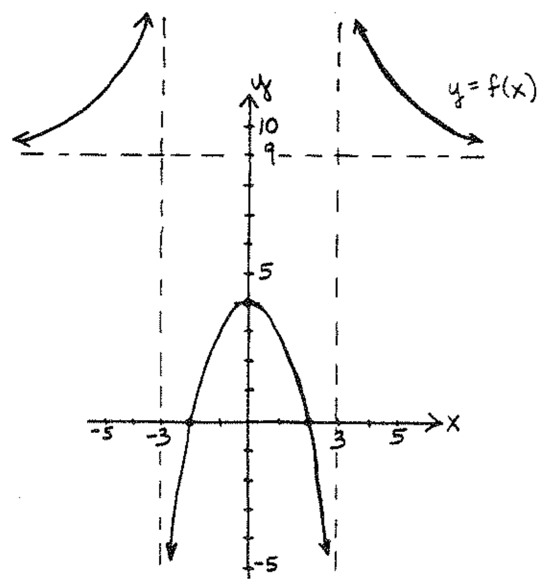
$f'(x) = \dfrac{-90x}{(x^2-9)^2}$

$f'(x)$: + + − − (sign chart on x with marks at -3, 0, 3)

graph of f: ↗ ↗ ↘ ↘

f is increasing on $(-\infty, -3)$ or $(-3, 0]$.

d.)



1986 - AB3, BC1

3. A particle moves along the x-axis so that at any time $t \geq 1$ its acceleration is given by $a(t) = \frac{1}{t}$. At time $t = 1$, the velocity of the particle is $v(1) = -2$ and its position is $x(1) = 4$.

(a) Find the velocity $v(t)$ for $t \geq 1$.

(b) Find the position $x(t)$ for $t \geq 1$.

(c) What is the position of the particle when it is farthest to the left?

---

a.) $v(t) = \int a(t)\,dt = \int \frac{1}{t}\,dt = \ln(t) + C_1, \quad t \geq 1$

$v(1) = -2 = \ln 1 + C_1 \Rightarrow C_1 = -2$

$v(t) = \ln(t) - 2, \quad t \geq 1$

b.) $x(t) = \int (\ln(t) - 2)\,dt$ | $u = \ln t$ $\quad dv = dt$

$= t\ln t - t - 2t + C_2$ | $du = \frac{dt}{t}$ $\quad v = t$

$x(1) = 4 = 1\ln 1 - 3 + C_2 \Rightarrow C_2 = 7$

$x(t) = t\ln t - 3t + 7, \quad t \geq 1$

c.) $x'(t) = v(t) = \ln t - 2$

$v(t) = 0$ when $\ln t = 2$ or $t = e^2$

Particle is furthest to the left when $t = e^2$

Position: $x(e^2) = 7 - e^2$

$v(t)$: $\quad -\quad +\quad \to t$; $\quad 1 \quad e^2$

Particle moves: left $\quad$ right

1986 - AB4

4. Let f be the function defined as follows.

$$f(x) = \begin{cases} |x-1|+2, & \text{for } x<1 \\ ax^2+bx, & \text{for } x \geqq 1, \text{ where } a \text{ and } b \text{ are constants.} \end{cases}$$

(a) If $a=2$ and $b=3$, is f continuous for all x? Justify your answer.

(b) Describe all values of a and b for which f is a continuous function.

(c) For what values of a and b is f both continuous and differentiable?

---

a.) If $a=2$ and $b=3$, then $f(x) = \begin{cases} -x+3, & \text{for } x<1 \\ 2x^2+3x, & \text{for } x \geqq 1 \end{cases}$

<u>for $x<1$: and $x>1$</u> $f(x)$ is a polynomial function and is therefore continuous.

<u>for $x=1$:</u> $f(1)=5$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(-x+3) = 2$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(2x^2+3x) = 5$$

$\lim_{x\to 1} f(x)$ does not exist since $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$.

$\therefore f(x)$ is continuous for all $x \neq 1$.

b.) For $f(x)$ continuous at $x=1$,

$$\lim_{x\to 1^-}(3-x) = \lim_{x\to 1^+}(ax^2+b)$$

$$2 = a+b$$

c.) $f'(x) = \begin{cases} -1 & \text{for } x<1 \\ 2ax+b & \text{for } x>1 \end{cases}$

$$\lim_{x\to 1^-}(-1) = \lim_{x\to 1^+}(2ax+b)$$

$$-1 = 2a+b$$

$$2 = a+b \quad \text{(part b)}$$

$a=-3$, $b=5$

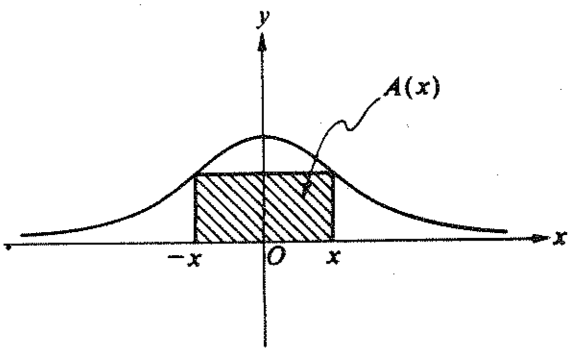
1986 - AB5, BC2

5. Let $A(x)$ be the area of the rectangle inscribed under the curve $y = e^{-2x^2}$ with vertices at $(-x, 0)$ and $(x, 0)$, $x \geqq 0$, as shown in the figure above.

(a) Find $A(1)$.

(b) What is the greatest value of $A(x)$? Justify your answer.

(c) What is the average value of $A(x)$ on the interval $0 \leqq x \leqq 2$?



a.) $A(x) = 2xy = 2xe^{-2x^2}, \quad x>0$

$A(1) = 2e^{-2} = \frac{2}{e^2}$

b.) $A'(x) = 2xe^{-2x^2}(-4x) + 2e^{-2x^2}$

$= 2e^{-2x^2}(-4x^2+1), \quad x>0$

$A'(x) \begin{cases} e^{-2x^2} : & + \quad + \\ (-4x^2+1) : & + \quad - \end{cases}$ on the x-axis with points 0 and $\frac{1}{2}$

Graph of A(x): ↗ ↘

By the First Derivative Test, A has an absol. max. at $x = \frac{1}{2}$.

$\boxed{\text{Absolute max of } A(x) = 2 \cdot \tfrac{1}{2} e^{-2(\frac{1}{4})} = e^{-\frac{1}{2}}}$

c.) $\text{Av } A(x) = \dfrac{\int_0^2 2xe^{-2x^2}dx}{2-0} = \dfrac{\int_0^2 -4xe^{-2x^2}dx}{-4}$

$= -\frac{1}{4}\left[e^{-2x^2}\right]_0^2 = \dfrac{1-e^{-8}}{4}$

$\boxed{\text{Average value of } A(x) \text{ on } [0,2] = \dfrac{1-e^{-8}}{4}}$

1986 - AB6, BC3

6. The shaded region R shown in the figure above is enclosed by the graphs of $y = \tan^2 x$, $y = \frac{1}{2}\sec^2 x$, and the y-axis.

(a) Find the area of region R.

(b) Set up, but do not integrate, an integral expression in terms of a single variable for the volume of the solid formed by revolving region R about the x-axis.

---

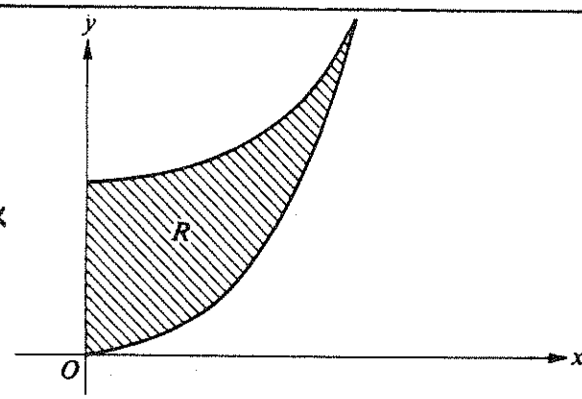
a.) $A = \int_0^{\pi/4} (y_1 - y_2)\,dx$

$= \int_0^{\pi/4} \left[\frac{\sec^2 x}{2} - (\sec^2 x - 1)\right] dx$

$= \int_0^{\pi/4} \left(1 - \frac{\sec^2 x}{2}\right) dx$

$= x - \frac{\tan x}{2} \Big]_0^{\pi/4}$

$= \pi/4 - \frac{1}{2}$



Point of Intersection:

$\tan^2 x = \frac{1 + \tan^2 x}{2}$

$\tan^2 x = 1$

$x = \pi/4$

b.) $dV = (\pi R^2 - \pi r^2)\,dx$

$V = \pi \int_0^{\pi/4} \left(\frac{\sec^4 x}{4} - \tan^4 x\right) dx$

1986 - BC4

4. Given the differential equation $\frac{dy}{dx} = 2y - 5\sin x$.

(a) Find the general solution.

(b) Find the particular solution whose tangent line at $x = 0$ has slope 7.

---

a.) Method 1 (Undetermined Coefficients)

General Solution: $y_G = y_H + y_P$ where

y_H is solution of $\frac{dy}{dx} - 2y = 0$ and $y_P = A\sin x + B\cos x$

$\frac{dy}{dx} = 2y \Rightarrow \frac{dy}{y} = 2\,dx \Rightarrow \ln y = 2x + C$

$\therefore\ y_H = Ce^{2x}$

$y_P = A\sin x + B\cos x$

$y'_P = A\cos x - B\sin x$

Substitute in given equation $\frac{dy}{dx} - 2y = -5\sin x$

$A\cos x - B\sin x - 2(A\sin x + B\cos x) = -5\sin x$

$(A - 2B)\cos x - (B + 2A)\sin x = -5\sin x$

$$\begin{array}{r} A - 2B = 0 \\ 2A + B = 5 \\ \hline 5A = 10 \end{array}$$

$A = 2,\ B = 1$

$\therefore\ y_P = 2\sin x + \cos x$

$y_G = Ce^{2x} + 2\sin x + \cos x$

Method 2

$$\frac{dy}{dx} - 2y = -5\sin x$$

Integrating Factor: $e^{\int -2\,dx} = e^{-2x}$

$$e^{-2x}\frac{dy}{dx} - 2ye^{-2x} = -5e^{-2x}\sin x \quad \underline{\text{OR}} \quad \frac{d(e^{-2x}y)}{dx} = -5e^{-2x}\sin x \Rightarrow$$

i.) $e^{-2x}y = -5\int e^{-2x}\sin x\,dx + C$

ii.) $\int e^{-2x}\sin x\,dx = -e^{-2x}\cos x - 2\int e^{-2x}\cos x\,dx$ | $u = e^{-2x}$, $dv = \sin x\,dx$, $du = -2e^{-2x}dx$, $v = -\cos x$

iii.) $\int e^{-2x}\cos x\,dx = e^{-2x}\sin x + 2\int e^{-2x}\sin x\,dx$ | $u = e^{-2x}$, $dv = \cos x\,dx$, $du = -2e^{-2x}dx$, $v = \sin x$

Substitute iii.) into ii.) :

$$\int e^{-2x}\sin x\,dx = -e^{-2x}\cos x - 2\left[e^{-2x}\sin x + 2\int e^{-2x}\sin x\,dx\right]$$

$$\int e^{-2x}\sin x\,dx = -e^{-2x}\cos x - 2e^{-2x}\sin x - 4\int e^{-2x}\sin x\,dx$$

iv.) $5\int e^{-2x}\sin x\,dx = -e^{-2x}\cos x - 2e^{-2x}\sin x$

Substitute iv.) into i.) :

$$e^{-2x}y = e^{-2x}\cos x + 2e^{-2x}\sin x + C$$

$$y = \cos x + 2\sin x + Ce^{2x}$$

b.) $\frac{dy}{dx} = -\sin x + 2\cos x + 2Ce^{2x}$

$\left.\frac{dy}{dx}\right|_{x=0} = 2 + 2C = 7 \Rightarrow C = \frac{5}{2}$

$y = \cos x + 2\sin x + \frac{5}{2}e^{2x}$

5. (a) Find the first four nonzero terms in the Taylor series expansion about $x = 0$ for $f(x) = \sqrt{1+x}$.

(b) Use the results found in part (a) to find the first four nonzero terms in the Taylor series expansion about $x = 0$ for $g(x) = \sqrt{1+x^3}$.

(c) Find the first four nonzero terms in the Taylor series expansion about $x = 0$ for the function h such that $h'(x) = \sqrt{1+x^3}$ and $h(0) = 4$.

---

a.) $f(x) = f(0) + f'(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \ldots$

$f(x) = (1+x)^{1/2} \Rightarrow f(0) = 1$

$f'(x) = \frac{1}{2}(1+x)^{-1/2} \Rightarrow f'(0) = \frac{1}{2}$

$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)(1+x)^{-3/2} \Rightarrow f''(0) = -\frac{1}{4}$

$f'''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(1+x)^{-5/2} \Rightarrow f'''(0) = \frac{3}{8}$

$f(x) \approx 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}$

b.) $g(x) = \sqrt{1+x^3} = f(x^3)$

$g(x) = f(x^3) \approx 1 + \frac{x^3}{2} - \frac{x^6}{8} + \frac{x^9}{16}$

c.) $h'(x) = f(x^3)$ or $h'(x) = g(x)$

$h(x) = \int g(x)\,dx$

$= \int \left(1 + \frac{x^3}{2} - \frac{x^6}{8} + \frac{x^9}{16} + \ldots\right) dx$

$= C + x + \frac{x^4}{8} - \frac{x^7}{56} + \frac{x^{10}}{160} + \ldots$

$h(0) = 4 = C$

$\therefore h(x) \approx 4 + x + \frac{x^4}{8} - \frac{x^7}{56}$

1986 - BC6

6. For all real numbers x and y, let f be a function such that $f(x+y) = f(x) + f(y) + 2xy$ and such that $\lim_{h \to 0} \frac{f(h)}{h} = 7$.

(a) Find $f(0)$. Justify your answer.

(b) Use the definition of the derivative to find $f'(x)$.

(c) Find $f(x)$.

---

a.) Let $x = y = 0$

$f(0+0) = f(0) + f(0) + 0$

$f(0) = 2f(0) \Rightarrow f(0) = 0$

OR Let $x = 0$

$f(0+y) = f(0) + f(y)$

$f(0) = 0$

b.) $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Let $y = h$. Then $f(x+h) - f(x) = f(h) + 2xh$

and $\frac{f(x+h) - f(x)}{h} = \frac{f(h) + 2xh}{h}$

$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{f(h)}{h} + \lim_{h \to 0} 2x$

$\boxed{f'(x) = 7 + 2x}$

c.) $f(x) = \int (7 + 2x)\,dx = 7x + x^2 + C$

Since $f(0) = 0$, then $C = 0$

$\therefore \boxed{f(x) = 7x + x^2}$

1. A particle moves along the x-axis so that its acceleration at any time t is given by $a(t) = 6t - 18$. At time $t = 0$ the velocity of the particle is $v(0) = 24$, and at time $t = 1$ its position is $x(1) = 20$.

   (a) Write an expression for the velocity $v(t)$ of the particle at any time t.

   (b) For what values of t is the particle at rest?

   (c) Write an expression for the position $x(t)$ of the particle at any time t.

   (d) Find the total distance traveled by the particle from $t = 1$ to $t = 3$.

---

a) $v(t) = \int a(t)\,dt = \int (6t-18)\,dt = 3t^2 - 18t + C_1$

$v(0) = 24 = 3(0)^2 - 18(0) + C_1 \Rightarrow C_1 = 24$

$\boxed{v(t) = 3t^2 - 18t + 24}$

b) Particle is at rest when $v(t) = 0$.

$3(t^2 - 6t + 8) = 0$

$3(t-4)(t-2) = 0$

$\boxed{t = 4,\ t = 2}$

c) $x(t) = \int v(t)\,dt = \int (3t^2 - 18t + 24)\,dt = t^3 - 9t^2 + 24t + C_2$

$x(1) = 20 = 1 - 9 + 24 + C_2 \Rightarrow C_2 = 4$

$\boxed{x(t) = t^3 - 9t^2 + 24t + 4}$

d) Method 1:

$x(1) = 1 - 9 + 24 + 4 = 20$ > Right 4

$x(2) = 8 - 36 + 48 + 4 = 24$ > Left 2

$x(3) = 27 - 81 + 72 + 4 = 22$

$\boxed{\text{Total Distance} = 6}$

Method 2: Total Distance $= \int_1^3 |v(t)|\,dt$

$= \int_1^2 v(t)\,dt - \int_2^3 v(t)\,dt = 6$

1987 – AB 2

2. Let $f(x) = \sqrt{1 - \sin x}$.

(a) What is the domain of f?

(b) Find $f'(x)$.

(c) What is the domain of f'?

(d) Write an equation for the line tangent to the graph of f at $x = 0$.

---

a.) $|\sin x| \leq 1 \;\; \forall x \;\Leftrightarrow\; 1 - \sin x \geq 0 \;\; \forall x$

$$\boxed{D_f = \text{set of all real numbers} \;\; \underline{\text{OR}} \;\; \{x \mid x \in \mathbb{R}\}}$$

b.) $f(x) = (1 - \sin x)^{1/2}$

$f'(x) = \frac{1}{2}(1 - \sin x)^{-\frac{1}{2}}(-\cos x)$

$$\boxed{f'(x) = \frac{-\cos x}{2\sqrt{1 - \sin x}}}$$

c.) $1 - \sin x \neq 0 \;\Rightarrow\; \sin x \neq 1$

$$\boxed{D_{f'} = \text{set of all real } x \neq \frac{\pi}{2} + 2\pi n, \; n = \text{any integer}}$$

d.) $f(0) = \sqrt{1 - \sin 0} = 1$

$f'(0) = \dfrac{-\cos 0}{2\sqrt{1 - \sin 0}} = -\dfrac{1}{2}$

$$\boxed{\begin{array}{c} y - 1 = -\frac{1}{2}(x - 0) \\ \underline{\text{OR}} \\ y = -\frac{1}{2}x + 1 \end{array}}$$

1987 – AB3

3. Let R be the region enclosed by the graphs of $y = (64x)^{\frac{1}{4}}$ and $y = x$.

   (a) Find the volume of the solid generated when region R is revolved about the $\underline{x\text{-axis}}$.

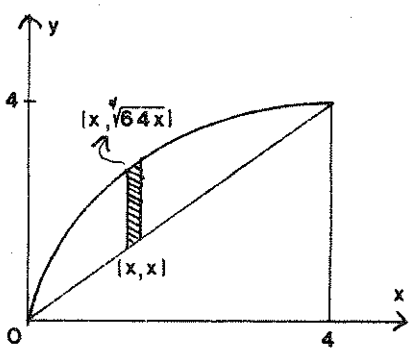
   (b) Set up, but do not integrate, an integral expression in terms of a single variable for the volume of the solid generated when region R is revolved about the $\underline{y\text{-axis}}$.

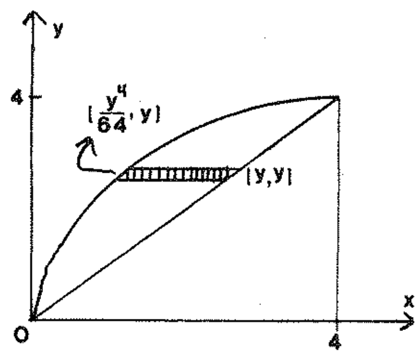
Points of intersection:

$$\sqrt[4]{64x} = x$$
$$64x = x^4$$
$$0 = x(x^3 - 64)$$
$$x = 0,\ x = 4$$

x as a function of y:

$$y = \sqrt[4]{64x} \iff x = \frac{y^4}{64}$$





a.) "Washers":

$$V_x = \pi \int_0^4 \left(\left[(64x)^{1/4} \right]^2 - (x)^2 \right) dx$$
$$= \pi \int_0^4 \left[(64x)^{1/2} - x^2 \right] dx$$
$$= \pi \int_0^4 \left(8x^{1/2} - x^2 \right) dx$$
$$= \pi \left(8 \cdot \frac{2}{3} x^{3/2} - \frac{x^3}{3} \right) \Big]_0^4$$
$$= \pi \left(\frac{16}{3} \cdot 8 - \frac{64}{3} \right)$$
$$= \frac{64\pi}{3}$$

OR

"Shells":

$$V_x = 2\pi \int_0^4 y \left(y - \frac{y^4}{64} \right) dy$$
$$= 2\pi \int_0^4 \left(y^2 - \frac{y^5}{64} \right) dy$$
$$= 2\pi \left(\frac{y^3}{3} - \frac{y^6}{64(6)} \right) \Big]_0^4$$
$$= 2\pi \left(\frac{64}{3} - \frac{64}{6} \right)$$
$$= 2\pi \left(\frac{32}{3} \right)$$
$$= \frac{64\pi}{3}$$

b) "Shells":

$$V_y = 2\pi \int_0^4 x \left[(64x)^{1/4} - x \right] dx$$

OR

"Washers":

$$V_y = \pi \int_0^4 \left[y^2 - \left(\frac{y^4}{64} \right)^2 \right] dy$$

1987 – AB 4

4. Let f be the function given by $f(x) = 2\ln(x^2+3) - x$ with domain $-3 \leqq x \leqq 5$.

(a) Find the x-coordinate of each relative maximum point and each relative minimum point of f. Justify your answer.

(b) Find the x-coordinate of each inflection point of f.

(c) Find the absolute maximum value of $f(x)$.

---

a.) $f'(x) = \dfrac{4x}{x^2+3} - 1 = \dfrac{-(x-3)(x-1)}{x^2+3}$; $f'(x)=0$ when $x = 1, 3$

$f'(x)$: on $[-3, 1]$: $-$; on $[1, 3]$: $+$; on $[3, 5]$: $-$

Graph of f: $\searrow$ $\nearrow$ $\searrow$

f has a rel. min. at $x=1$ and a rel. max. at $x=3$

b.) $f''(x) = \dfrac{(x^2+3)4 - 4x(2x)}{(x^2+3)^2} = \dfrac{4(3-x^2)}{(x^2+3)^2}$

$f''(x) = 0$ when $x = \pm\sqrt{3}$

$f''(x)$: on $[-3, -\sqrt{3}]$: $-$; on $[-\sqrt{3}, \sqrt{3}]$: $+$; on $[\sqrt{3}, 5]$: $-$

Concavity of f: d | u | d

f has inflection points at $x = \pm\sqrt{3}$

c.) Check $f(-3)$ and $f(3)$, since $f(1)$ is a rel. min. value of f and $f(5) < f(3)$.

$f(-3) = 2\ln 12 + 3$

$f(3) = 2\ln 12 - 3$

The absolute maximum value of $f(x) = 2\ln 12 + 3$

1987 – AB5

5. The trough shown in the figure above is 5 feet long, and its vertical cross sections are inverted isosceles triangles with base 2 feet and height 3 feet. Water is being siphoned out of the trough at the rate of 2 cubic feet per minute. At any time t, let h be the depth and V be the volume of water in the trough.

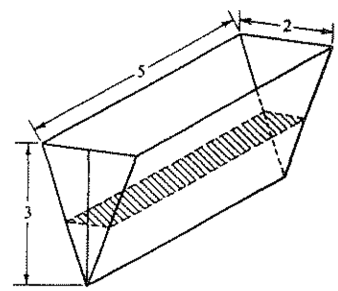
(a) Find the volume of water in the trough when it is full.

(b) What is the rate of change in h at the instant when the trough is $\frac{1}{4}$ full by volume?

(c) What is the rate of change in the area of the surface of the water (shaded in the figure) at the instant when the trough is $\frac{1}{4}$ full by volume?

Given: $\frac{dV}{dt} = -2$

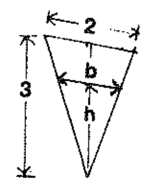
a) $V = \frac{1}{2} \times 2 \times 3 \times 5 = 15 \text{ ft}^3$



b) Find $\frac{dh}{dt}$ when $V = \frac{15}{4}$

$V = \frac{bh}{2} \times 5$; $\frac{b}{h} = \frac{2}{3} \Rightarrow b = \frac{2h}{3}$

$V = \frac{5}{3} h^2$



When $V = \frac{15}{4}$, $h^2 = \frac{9}{4}$ and $h = \frac{3}{2}$

$\frac{dV}{dt} = \frac{10h}{3} \frac{dh}{dt}$

$-2 = \frac{10}{3}\left(\frac{3}{2}\right)\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = -\frac{2}{5}$

$\boxed{h \text{ is changing at } -\frac{2}{5} \text{ ft./min. when } h = \frac{3}{2} \text{ ft.}}$

c) Find $\frac{dA}{dt}$ when $V = \frac{15}{4}$ or $h = \frac{3}{2}$

$A = 5b = \frac{10}{3} h$

$\frac{dA}{dt} = \frac{10}{3}\frac{dh}{dt} = \frac{10}{3}\left(-\frac{2}{5}\right) = -\frac{4}{3}$

$\boxed{\text{Area is changing at } -\frac{4}{3} \text{ sq.ft./min. when } V = \frac{15}{4} \text{ ft.}^3}$

1987 – AB6

6. Let f be a function such that $f(x) < 1$ and $f'(x) < 0$ for all x.

(a) Suppose that $f(b) = 0$ and $a < b < c$. Write an expression involving integrals for the area of the region enclosed by the graph of f, the lines $x = a$ and $x = c$, and the x-axis.

(b) Determine whether $g(x) = \dfrac{1}{f(x) - 1}$ is increasing or decreasing. Justify your answer.

(c) Let h be a differentiable function such that $h'(x) < 0$ for all x. Determine whether $F(x) = h(f(x))$ is increasing or decreasing. Justify your answer.

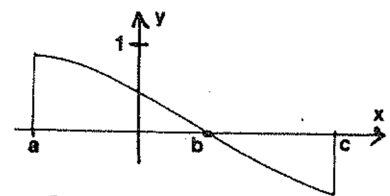
---

a) Given: $f(x) < 1,\ f'(x) < 0\ \forall x,\ f(b) = 0,\ a < b < c$

$f'(x) < 0 \Rightarrow f$ is strictly decreasing on $[a, c]$

$$A = \int_a^c |f(x)|\, dx$$

$$A = \int_a^b f(x)\, dx - \int_b^c f(x)\, dx$$



b) $g(x) = (f(x) - 1)^{-1} \Rightarrow g'(x) = -(f(x)-1)^{-2}(f'(x))$

Since $f(x) < 1$, $-(f(x) - 1)^{-2} < 0$, and $f'(x) < 0$, $g'(x) > 0$

$\therefore$ g is increasing

OR $f'(x) < 0 \Rightarrow f(x)$ decr. $\forall x \Rightarrow f(x) - 1$ decr. $\forall x$

Since $f(x) < 1$, $f(x) - 1 < 0\ \forall x$. Then

$\dfrac{1}{f(x) - 1} = g(x)$ is increasing $\forall x$

c) $F(x) = h(f(x)) \Rightarrow F'(x) = h'(f(x)) \cdot f'(x)$

Since $h'(x) < 0$ and $f'(x) < 0\ \forall x$, $F'(x) > 0$

$\therefore$ F is increasing

OR $f'(x) < 0 \Rightarrow f$ decreasing $\forall x$

Then $f(x_1) > f(x_2)$ when $x_1 < x_2$

$h'(x) < 0 \Rightarrow h$ decreasing $\forall x$

Then $h(f(x_1)) < h(f(x_2)) \Rightarrow F$ is increasing

1987 - BC1

1. At any time $t \geq 0$, in days, the rate of growth of a bacteria population is given by $y' = ky$, where k is a constant and y is the number of bacteria present. The initial population is 1,000 and the population triples during the first 5 days.

   (a) Write an expression for y at any time $t \geq 0$.

   (b) By what factor will the population have increased in the first 10 days?

   (c) At what time t, in days, will the population have increased by a factor of 6 ?

Given: $\frac{dy}{dt} = ky$, $t \geq 0$; $y_0 = 1000$, $y_5 = 3000$

a) $\frac{dy}{dt} = kt \Leftrightarrow y_t = y_0 e^{kt}$

$y_5 = 3000 = 1000e^{5k}$

$e^{5k} = 3 \Rightarrow 5k = \ln 3$, $k = \frac{\ln 3}{5}$

$$\boxed{y_t = 1000e^{\frac{t\ln 3}{5}}, \quad \text{OR} \quad y_t = 1000 \cdot 3^{t/5}}$$

$\frac{dy}{dt} = ky \Rightarrow \frac{dy}{y} = k\,dt$

$\int \frac{dy}{y} = \int k\,dt \Rightarrow \ln y = kt + C_1$

OR $y = e^{kt + C_1} \Leftrightarrow y = C_2 e^{kt}$

$t = 0 \Rightarrow y_0 = C_2$

$y_t = y_0 e^{kt}$

b) $y_{10} = 1000e^{\frac{10\ln 3}{5}}$

$= 1000 \cdot 3^2 = 9000$

$$\boxed{\text{Population increased by a factor of 9 in first 10 days}}$$

c) $6000 = 1000e^{\frac{t\ln 3}{5}}$

$\ln 6 = \frac{t\ln 3}{5} \Rightarrow t = \frac{5\ln 6}{\ln 3}$ or $5\log_3 6$

$$\boxed{\text{Population increases by a factor of 6 in } \frac{5\ln 6}{\ln 3} \text{ or } 5\log_3 6 \text{ days}}$$

1987 – BC 2

2. Consider the curve given by the equation $y^3 + 3x^2y + 13 = 0$.

(a) Find $\frac{dy}{dx}$.

(b) Write an equation for the line tangent to the curve at the point $(2, -1)$.

(c) Find the minimum y-coordinate of any point on the curve. Justify your answer.

a.) $3y^2\frac{dy}{dx} + 3\left(x^2\frac{dy}{dx} + 2xy\right) = 0$

$$\boxed{\frac{dy}{dx} = \frac{-2xy}{x^2+y^2}}$$

b.) $y - y_1 = m(x - x_1)$ where $(x_1, y_1) = (2, -1)$ and

$$m = \left.\frac{dy}{dx}\right|_{(2,-1)} = \frac{-2(2)(-1)}{5} = \frac{4}{5}$$

$$\boxed{y + 1 = \frac{4}{5}(x-2)}.$$

c.) $\frac{dy}{dx} = 0$ when $x = 0$; $x = 0 \Rightarrow y = \sqrt[3]{-13}$

$y < 0 \ \forall x$ since $y = \dfrac{-13}{3x^2+y^2}$

$\frac{dy}{dx}$: $-$ $\;0\;$ $+$ $\to X$

Graph of y: $\searrow$ $\;0\;$ $\nearrow$

$\therefore$ y has an absolute min. of $\sqrt[3]{-13}$ when $x = 0$

<u>Alternate Methods</u>:

$y^3 + 13 = -3x^2y$

Since $y < 0$, $y^3 + 13 \geq 0$

or $y \geq -\sqrt[3]{13}$

When $x = 0$, $y = -\sqrt[3]{13}$

$\therefore$ $y = -\sqrt[3]{13}$ is a min.

(By contradiction) $x = 0 \Rightarrow y = -\sqrt[3]{13}$

Suppose $\exists x$ such that $y < -\sqrt[3]{13}$. Then $y^3 < -13$. Let $y^3 = -13 - h$, $h > 0$.

Substitute in orig. eq.: $-13 - h + 3x^2y + 13 = 0$ or $3x^2y = h > 0$. But $y < 0 \Rightarrow 3x^2y < 0$.

This contradiction makes the original assumption false. $\therefore$ y min $= -\sqrt[3]{13}$

1987 - BC 3

3. Let R be the region enclosed by the graph of $y = \ln x$, the line $x = 3$, and the x-axis.

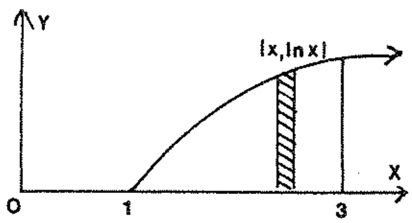
(a) Find the area of region R.

(b) Find the volume of the solid generated by revolving region R about the x-axis.

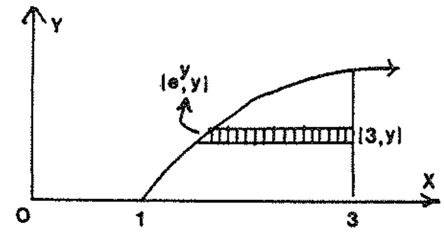
(c) Set up, but <u>do not integrate</u>, an integral expression in terms of a single variable for the volume of the solid generated by revolving region R about the line $x = 3$.

---

a.)



OR



$$A = \int_1^3 \ln x \, dx$$

$$A = x\ln x - x\Big]_1^3 \quad \text{Method of Parts (see below)}$$

$$A = 3\ln 3 - 2.$$

OR

$$y = \ln x \Leftrightarrow x = e^y, \; x > 0$$

$$A = \int_0^{\ln 3} (3 - e^y)\, dy = 3y - e^y\Big]_0^{\ln 3}$$

$$A = 3\ln 3 - 3 + 1 = 3\ln 3 - 2$$

b.) "Discs":

$$V_x = \int_1^3 \pi (\ln x)^2 \, dx$$

Method of Parts (see below)

$$V_x = \pi\left[x(\ln x)^2 - 2(x\ln x - x)\right]_1^3$$

$$V_x = \pi\left[3(\ln 3)^2 - 6\ln 3 + 4\right]$$

OR "Shells":

$$V_x = \int_0^{\ln 3} 2\pi y (3 - e^y)\, dy$$

Method of Parts (see below)

$$V_x = 2\pi\left[\frac{3y^2}{2} - ye^y + e^y\right]_0^{\ln 3}$$

$$V_x = 2\pi\left[\frac{3(\ln 3)^2}{2} - 3\ln 3 + 2\right]$$

c.) "Shells":

$$V_y = 2\pi \int_1^3 (3 - x)\ln x \, dx$$

OR "Discs":

$$V_y = \pi \int_0^{\ln 3} (3 - e^y)^2 \, dy$$

<u>Method of Parts</u>

$\int \ln x \, dx$:

$u = \ln x \qquad dv = dx$

$du = \frac{dx}{x} \qquad v = x$

$$\int \ln x \, dx = x\ln x - \int dx$$
$$= x\ln x - x$$

$\int (\ln x)^2 dx$:

$u = (\ln x)^2 \qquad dv = dx$

$du = \frac{2\ln x}{x}dx \qquad v = x$

$$\int (\ln x)^2 dx = x(\ln x)^2 - \int 2\ln x \, dx$$
$$= x(\ln x)^2 - 2(x\ln x - x)$$

$\int y e^y dy$:

$u = y \qquad dv = e^y dy$

$du = dy \qquad v = e^y$

$$\int ye^y dy = ye^y - \int e^y dy$$
$$= ye^y - e^y$$

1987 – BC 4

4. (a) Find the first five terms in the Taylor series about $x = 0$ for $f(x) = \frac{1}{1-2x}$.

(b) Find the interval of convergence for the series in part (a).

(c) Use partial fractions and the result from part (a) to find the first five terms in the Taylor series about $x = 0$ for $g(x) = \frac{1}{(1-2x)(1-x)}$.

---

a) $f(x) = (1-2x)^{-1}$ $\quad f(0) = 1$ $\quad f'''(x) = -3!(1-2x)^{-4}(-2)^3$ $\quad f'''(0) = 3!\,2^3$

$f'(x) = -1(1-2x)^{-2}(-2)$ $\quad f'(0) = 2$ $\quad f^{IV}(x) = 4!(1-2x)^{-5}(-2)^4$ $\quad f^{IV}(0) = 4!\,2^4$

$f''(x) = 2!(1-2x)^{-3}(-2)^2$ $\quad f''(0) = 2!\,2^2$

$$f(x) = f(0) + f'(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + f^{IV}(0)\frac{x^4}{4!} + \dots$$

$$f(x) = 1 + 2x + 2^2x^2 + 2^3x^3 + \dots = \sum_0^{\infty}(2x)^n$$

$$f(x) \approx 1 + 2x + 4x^2 + 8x^3 + 16x^4$$

Alt. Method by "Long Division": $\frac{1}{1-2x} = 1 + 2x + 4x^2 + 8x^3 + 16x^4 + \dots$

b) Ratio Test: $\rho = \lim_{n\to\infty}\left|\frac{u^{n+1}}{u^n}\right| = \lim_{n\to\infty}\left|\frac{(2x)^n}{(2x)^{n-1}}\right| = |2x|$

Series converges if $|2x| < 1$ or $-\frac{1}{2} < x < \frac{1}{2}$

Endpoints: $x = \frac{1}{2} \Rightarrow f(x) = 1 + 1 + 1 + 1 + \dots$ which diverges

$x = -\frac{1}{2} \Rightarrow f(x) = 1 - 1 + 1 - 1 + \dots$ which diverges

$$\boxed{\text{Interval of Convergence: } -\tfrac{1}{2} < x < \tfrac{1}{2}}$$

Alternate Method: $\frac{1}{1-2x}$ is a geometric series which converges for $|2x| < 1$ or $-\frac{1}{2} < x < \frac{1}{2}$.

c) $g(x) = \frac{1}{(1-2x)(1-x)} = \frac{A}{1-2x} + \frac{B}{1-x} = \frac{A(1-x) + B(1-2x)}{(1-2x)(1-x)}$

$$\frac{1}{(1-2x)(1-x)} = \frac{-(A+2B)x + (A+B)}{(1-2x)(1-x)} \Rightarrow \left.\begin{matrix} A + 2B = 0 \\ A + B = 1 \end{matrix}\right\} \Rightarrow B = -1,\ A = 2$$

$$g(x) = \frac{2}{1-2x} - \frac{1}{1-x}$$

Replace $2x$ in series (a) by x: $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$

Then $g(x) = 2(1 + 2x + 4x^2 + 8x^3 + 16x^4 + \dots) - (1 + x + x^2 + x^3 + x^4 + \dots)$

$$\boxed{g(x) \approx 1 + 3x + 7x^2 + 15x^3 + 31x^4}$$

5. The position of a particle moving in the xy-plane at any time t, $0 \leq t \leq 2\pi$, is given by the parametric equations $x = \sin t$ and $y = \cos(2t)$.

(a) Find the velocity vector for the particle at any time t, $0 \leq t \leq 2\pi$.

(b) For what values of t is the particle at rest?

(c) Write an equation for the path of the particle in terms of x and y that does not involve trigonometric functions.

(d) Sketch the path of the particle in the xy-plane below.

---

a.) $\frac{dx}{dt} = \cos t$, $\frac{dy}{dt} = -2\sin(2t)$, $0 \leq t \leq 2\pi$

$\vec{v}_t = (\cos t, -2\sin(2t))$ or $\vec{v}_t = \cos t\,\vec{i} - 2\sin(2t)\vec{j}$

b.) particle is at rest when $\vec{v}_t = 0$

$\vec{v}_t = 0$ when $\cos t = 0$ and $-2\sin(2t) = 0$

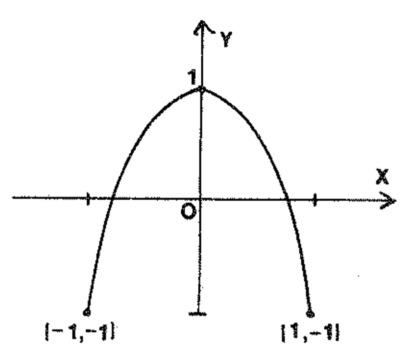
or when $\cos t = 0$ and $-4\sin t \cos t = 0$

$\boxed{\therefore\ \vec{v}_t = 0 \text{ when } \cos t = 0 \text{ or } t = \pi/2,\ 3\pi/2}$

c.) $y = \cos(2t) = \cos^2 t - \sin^2 t = 1 - 2\sin^2 t$

$\boxed{y = 1 - 2x^2}$

d.)



1987 - BC 6

6. Let f be a continuous function with domain $x > 0$ and let F be the function given by $F(x) = \int_1^x f(t)dt$ for $x > 0$. Suppose that $F(ab) = F(a) + F(b)$ for all $a > 0$ and $b > 0$ and that $F'(1) = 3$.

(a) Find $f(1)$.

(b) Prove that $aF'(ax) = F'(x)$ for every positive constant a.

(c) Use the results from parts (a) and (b) to find $f(x)$. Justify your answer.

a.) $F(x) = \int_1^x f(t)dt, \ x > 0 \Rightarrow F'(x) = f(x)$

$$F'(1) = 3 = f(1)$$

b.) $F(ax) = F(a) + F(x)$, $\underline{a}$ pos. const., $x > 0$

$$\frac{d(F(ax))}{dx} = \frac{d(F(a) + F(x))}{dx}$$

$$aF'(ax) = F'(x), \quad \underline{a} \text{ pos. const.}$$

c.) Let $x = 1$. Then

$$aF'(a) = F'(1) = 3, \ \forall a > 0$$

$$F'(a) = \frac{3}{a}, \quad \forall a > 0$$

$$F'(x) = \frac{3}{x}, \quad \forall x > 0$$

$$\boxed{f(x) = \frac{3}{x}, \quad \forall x > 0}$$ (Subst. from (a.))

1988 - AB 1

1. Let f be the function given by $f(x) = \sqrt{x^4 - 16x^2}$.

(a) Find the domain of f.

(b) Describe the symmetry, if any, of the graph of f.

(c) Find $f'(x)$.

(d) Find the slope of the line normal to the graph of f at $x = 5$.

---

a.) $x^4 - 16x^2 \geq 0$ or $x^2(x^2-16) \geq 0$

$D_f: \; x \geq 4, \; x \leq -4, \; x = 0$

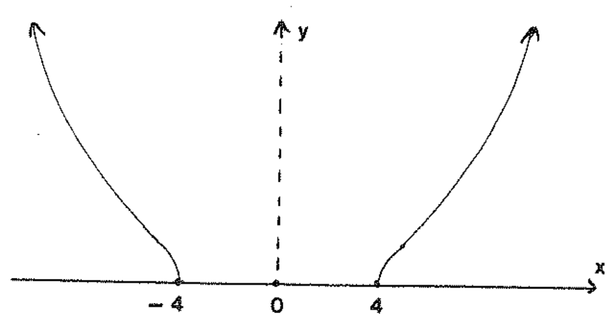
b.) $f(-x) = f(x) \Rightarrow$

Graph of f is symmetric with respect to the y-axis

c.) $f'(x) = \dfrac{4x^3 - 32x}{2\sqrt{x^4 - 16x^2}} = \dfrac{2x^3 - 16x}{\sqrt{x^2(x^2-16)}}$

d.) $f'(5) = \dfrac{170}{15} = \dfrac{34}{3}$

Slope of normal $= -\dfrac{1}{f'(5)} = -\dfrac{3}{34}$



1988 - AB 2

2. A particle moves along the x-axis so that its velocity at any time $t \geqq 0$ is given by $v(t) = 1 - \sin(2\pi t)$.

(a) Find the acceleration $a(t)$ of the particle at any time t.

(b) Find all values of t, $0 \leqq t \leqq 2$, for which the particle is at rest.

(c) Find the position $x(t)$ of the particle at any time t if $x(0) = 0$.

---

a.) $a(t) = -2\pi \cos 2\pi t, \quad t \geq 0$

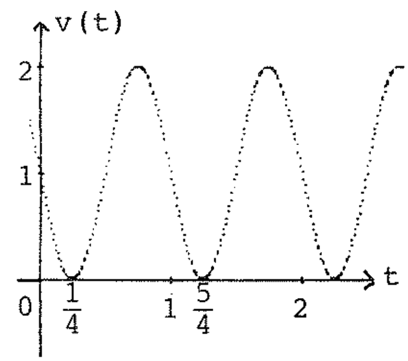
b.) Particle at rest when $v(t) = 0$

$v(t) = 0$ when $\sin 2\pi t = 1 \Rightarrow$

$2\pi t = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots \Rightarrow$

$t = \frac{1}{4}, \frac{5}{4}, \frac{9}{4}, \ldots$

Since $0 \leq t \leq 2$, $t = \frac{1}{4}$ or $\frac{5}{4}$



c.) $x(t) = \int v\,dt = \int (1 - \sin 2\pi t)\,dt$

$x(t) = t + \frac{\cos 2\pi t}{2\pi} + C$

$x(0) = 0 = 0 + \frac{1}{2\pi} + C \Rightarrow C = -\frac{1}{2\pi}$

$x(t) = t + \frac{\cos 2\pi t}{2\pi} - \frac{1}{2\pi}$

1988 - AB3

3. Let R be the region in the first quadrant enclosed by the hyperbola $x^2 - y^2 = 9$, the x-axis, and the line $x = 5$.

(a) Find the volume of the solid generated by revolving R about the x-axis.

(b) Set up, but do not integrate, an integral expression in terms of a single variable for the volume of the solid generated when R is revolved about the line $x = -1$.

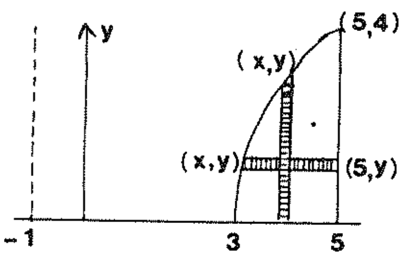
---

a.) "Disks":

$dV = \pi r^2 dx = \pi(x^2-9)\,dx$

$V_x = \pi\int_3^5 (x^2-9)dx = \pi\left[\frac{x^3}{3} - 9x\right]_3^5$

$V_x = \pi\left[\left(\frac{125}{3} - 45\right) - \left(\frac{27}{3} - 27\right)\right] = \frac{44\pi}{3}$



OR "Shells":

$dV = 2\pi r h\,dy = 2\pi y(5-x)dy$

$V_x = 2\pi\int_0^4 y\left(5 - \sqrt{9+y^2}\right)dy = 2\pi\left[\frac{5y^2}{2} - \frac{(9+y^2)^{3/2}}{3}\right]_0^4$

$V_x = 2\pi\left[\left(40 - \frac{125}{3}\right) + \frac{27}{3}\right] = \frac{44\pi}{3}$

b.) "Shells":

$dV = 2\pi r h\,dx$

$V_{x=-1} = 2\pi\int_3^5 (x+1)\sqrt{x^2-9}\,dx$

OR "Washers":

$dV = \pi(R^2 - r^2)dy$

$V_{x=-1} = \pi\int_0^4 \left(6^2 - (x+1)^2\right)dy$

$V_{x=-1} = \pi\int_0^4 \left(36 - \left(\sqrt{y^2+9}+1\right)^2\right)dy$

1988 – AB4

4. Let f be the function defined by $f(x) = 2xe^{-x}$ for all real numbers x.

(a) Write an equation of the horizontal asymptote for the graph of f.

(b) Find the x-coordinate of each critical point of f. For each such x, determine whether $f(x)$ is a relative maximum, a relative minimum, or neither.

(c) For what values of x is the graph of f concave down?

(d) Using the results found in parts (a), (b), and (c), sketch the graph of $y = f(x)$ in the xy-plane provided below. Note: The xy-plane is provided in the pink test booklet only.

---

a.) By l'Hôpital's Rule: $\lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0$

Equation of horizontal asymptote: $y = 0$

b.) $f'(x) = -2xe^{-x} + 2e^{-x} = 2e^{-x}(1-x)$

$f'(x)$: $+$ on $x < 1$, $-$ on $x > 1$ (sign change at 1)

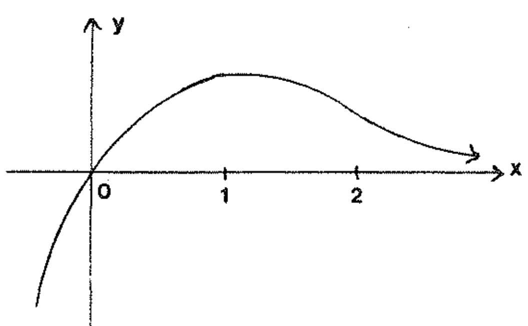
At $x = 1$, a critical point exists. $f(1)$ is a relative max (also an absolute max).

c.) $f''(x) = -2e^{-x} + (1-x)(-2e^{-x}) = -2e^{-x}(2-x)$

$f''(x)$: $-$ on $x < 2$, $+$ on $x > 2$ (sign change at 2)

Graph of f is concave down $\forall x < 2$

d.)



1988 - AB5

5. Let R be the region in the first quadrant under the graph of $y = \dfrac{x}{x^2 + 2}$ for $0 \leqq x \leqq \sqrt{6}$.

(a) Find the area of R.

(b) If the line $x = k$ divides R into two regions of equal area, what is the value of k?

(c) What is the average value of $y = \dfrac{x}{x^2 + 2}$ on the interval $0 \leqq x \leqq \sqrt{6}$?

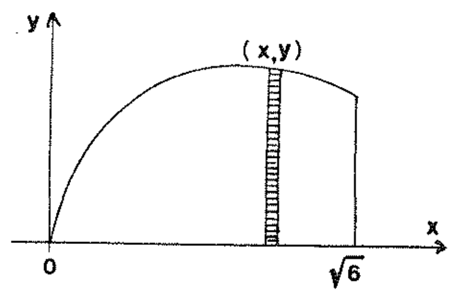
---

a.) $dA = y\,dx = \dfrac{x}{x^2+2}\,dx$

$$A = \int_0^{\sqrt{6}} \frac{x}{x^2+2}\,dx \qquad \text{Let } u = x^2+2, \quad du = 2x\,dx$$

$$= \frac{1}{2} \ln (x^2+2) \Big]_0^{\sqrt{6}}$$

$$= \frac{1}{2}\left[\ln 8 - \ln 2\right] = \ln 2$$



b.) $\displaystyle\int_0^{k} \frac{x}{x^2+2}\,dx = \frac{1}{2}\ln 2, \qquad 0 < k < \sqrt{6}$

$$\frac{1}{2}\left(\ln (k^2+2) - \ln 2\right) = \frac{1}{2}\ln 2$$

$$\ln\left(\frac{k^2+2}{2}\right) = \ln 2 \Rightarrow \frac{k^2+2}{2} = 2$$

$$k = \sqrt{2}$$

c.) $$y_{av} = \frac{\int_0^{\sqrt{6}} \frac{x}{x^2+2}\,dx}{\sqrt{6}} = \frac{\ln 2}{\sqrt{6}}$$

1988 - AB 6

6. Let f be a differentiable function, defined for all real numbers x, with the following properties.

(i) $f'(x) = ax^2 + bx$

(ii) $f'(1) = 6$ and $f''(1) = 18$

(iii) $\int_1^2 f(x)dx = 18$

Find $f(x)$. Show your work.

$f'(x) = ax^2 + bx \Rightarrow f''(x) = 2ax + b$

$\left.\begin{aligned} f'(1) &= a+b = 6 \\ f''(1) &= 2a+b = 18 \end{aligned}\right\} \Rightarrow a = 12, \ b = -6$

$f'(x) = 12x^2 - 6x$

$f(x) = \int (12x^2 - 6x)\,dx = 4x^3 - 3x^2 + C$

$\int_1^2 (4x^3 - 3x^2 + C)\,dx = 18$

$\left[(x^4 - x^3 + Cx)\right]_1^2 = 18$

$\left[(16 - 8 + 2C) - (1 - 1 + C)\right] = 18$

$8 + C = 18$

$C = 10$

$\boxed{f(x) = 4x^3 - 3x^2 + 10}$

1988 - BC1

1. Let f be the function defined by $f(x) = (x^2 - 3)\,e^x$ for all real numbers x.

(a) For what values of x is f increasing?
(b) Find the x-coordinate of each point of inflection of f.
(c) Find the x- and y- coordinates of the point, if any, where $f(x)$ attains its absolute minimum.

---

a.) $f'(x) = (x^2-3)e^x + 2xe^x = e^x(x+3)(x-1)$

	$x<-3$	$-3<x<1$	$x>1$
$x+3$:	$-$	$+$	$+$
$x-1$:	$-$	$-$	$+$
$f'(x)$:	$+$	$-$	$+$

f is increasing when $x \le -3$ or $x \ge 1$

b.) $f''(x) = e^x(2x+2) + e^x(x^2+2x-3) = e^x(x^2+4x-1)$

$x^2+4x-1=0$ when $x = \dfrac{-4 \pm \sqrt{20}}{2}$ or $-2 \pm \sqrt{5}$

	$x<-2-\sqrt{5}$	$-2-\sqrt{5}<x<-2+\sqrt{5}$	$x>-2+\sqrt{5}$
$f''(x)$:	$+$	$-$	$+$

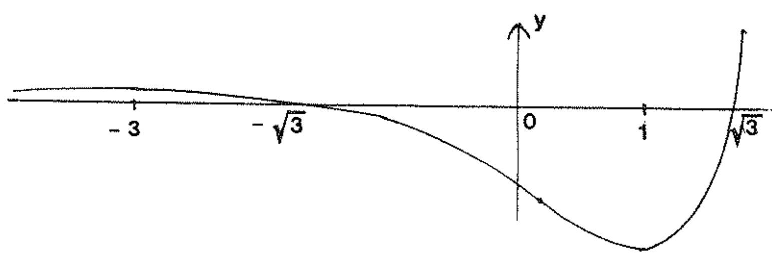
f has inflection points at $x = -2 \pm \sqrt{5}$

c.) By First Derivative Test (see (a)), f has an absolute min. at $(1, -2e)$ on $[-3, \infty)$. Since $f(x) = (x^2-3)e^x > 0$ $\forall x < -3$, f has an absolute min. at $(1, -2e)$.

OR $f(x)$ has zeros at $x = \pm\sqrt{3}$ and a rel. min. at $x = 1$

	$x<-\sqrt{3}$	$-\sqrt{3}<x<\sqrt{3}$	$x>\sqrt{3}$
$f(x)$:	$+$	$-$	$+$

$\therefore$ f has an absolute min. at $(1, -2e)$



1988 – BC2

2. Let R be the shaded region between the graphs of $y = \frac{3}{x}$ and $y = \frac{3x}{x^2+1}$ from $x = 1$ to $x = \sqrt{3}$, as shown in the figure above.

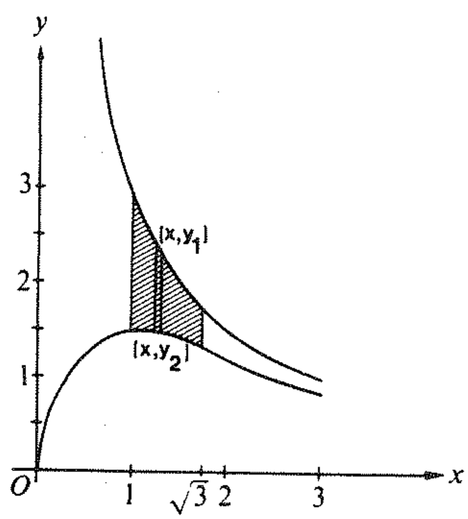
(a) Find the area of R.
(b) Find the volume of the solid generated by revolving R about the y-axis.

a.) $dA = (y_1 - y_2)\,dx$

$$A = \int_1^{\sqrt{3}} \left(\frac{3}{x} - \frac{3x}{x^2+1}\right)dx$$
$$= \left[3\ln x - \frac{3}{2}\ln(x^2+1)\right]_1^{\sqrt{3}}$$
$$= \left(3\ln\sqrt{3} - \frac{3}{2}\ln 4\right) - \left(3\ln 1 - \frac{3}{2}\ln 2\right)$$
$$= \frac{3}{2}\ln 3 - 3\ln 2 + \frac{3}{2}\ln 2 = \frac{3}{2}(\ln 3 - \ln 2)$$
$$= \frac{3}{2}\ln\left(\frac{3}{2}\right)$$

b.) "Shells":

$$dV_y = 2\pi r h\,dx = 2\pi x(y_1 - y_2)\,dx$$
$$V_y = 2\pi\int_1^{\sqrt{3}} x\left(\frac{3}{x} - \frac{3x}{x^2+1}\right)dx = 6\pi\int_1^{\sqrt{3}}\left(1 - \frac{x^2}{x^2+1}\right)dx$$
$$= 6\pi\int_1^{\sqrt{3}} \frac{1}{x^2+1}\,dx = 6\pi \operatorname{Arctan} x\Big]_1^{\sqrt{3}}$$
$$= 6\pi\left[\operatorname{Arctan}\sqrt{3} - \operatorname{Arctan} 1\right] = 6\pi\left[\frac{\pi}{3} - \frac{\pi}{4}\right]$$
$$V_y = \frac{\pi^2}{2}$$

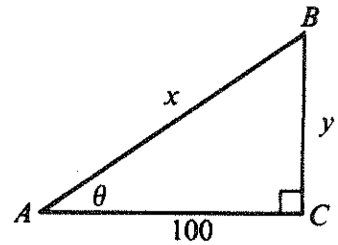


1988 BC2

1988 - BC3

3. The figure above represents an observer at point A watching balloon B as it rises from point C. The balloon is rising at a constant rate of 3 meters per second and the observer is 100 meters from point C.

(a) Find the rate of change in x at the instant when $y = 50$.
(b) Find the rate of change in the area of right triangle BCA at the instant when $y = 50$.
(c) Find the rate of change in θ at the instant when $y = 50$.



a) GIVEN: $\frac{dy}{dt} = 3$

FIND: $\frac{dx}{dt}$ when $y = 50$

$x^2 = y^2 + 100^2 \Rightarrow \cancel{2}x\frac{dx}{dt} = \cancel{2}y\frac{dy}{dt}$

$x^2 = 50^2 + 100^2 = 50^2(1+4)$

$x = 50\sqrt{5}$

$\frac{dx}{dt} \cdot 50\sqrt{5} = 50(3)$

$\frac{dx}{dt} = \frac{3}{\sqrt{5}}$

$\boxed{x \text{ is changing at the rate of } \frac{3}{\sqrt{5}} \text{ m/sec when } y = 50 \text{ m.}}$

b) Find $\frac{dA}{dt}$ when $y = 50$

$A = 50y \Rightarrow \frac{dA}{dt} = 50\frac{dy}{dt} \Rightarrow \left.\frac{dA}{dt}\right|_{y=50} = 50 \cdot 3 = 150$

$\boxed{\text{Area is changing at the rate of } 150 \text{ m}^2\text{/sec when } y = 50 \text{ m.}}$

c) Find $\frac{d\theta}{dt}$ when $y = 50$; $x = 50\sqrt{5}$

$\tan\theta = \frac{y}{100}$

$\sec^2\theta\frac{d\theta}{dt} = \frac{1}{100}\frac{dy}{dt}$

$\frac{d\theta}{dt} = \cos^2\theta \cdot \frac{3}{100}$

$\left.\frac{d\theta}{dt}\right|_{y=50} = \frac{4}{5} \cdot \frac{3}{100} = \frac{3}{125}$

OR

$\theta = \text{Arctan}\frac{y}{100}$

$\frac{d\theta}{dt} = \frac{\frac{1}{100}\frac{dy}{dt}}{1 + \left(\frac{y}{100}\right)^2}$

$\left.\frac{d\theta}{dt}\right|_{y=50} = \frac{\frac{3}{100}}{1 + \frac{1}{4}} = \frac{3}{125}$

$\boxed{\theta \text{ is changing at the rate of } \frac{3}{125} \text{ rad/sec when } y = 50 \text{ m.}}$

1988 – BC4

4. Determine all values of x for which the series $\sum_{k=0}^{\infty} \frac{2^k x^k}{\ln(k+2)}$ converges. Justify your answer.

Series converges absolutely if $\lim_{k\to\infty}\left|\frac{u_{k+1}}{u_k}\right| < 1$

$$\left|\frac{u_{k+1}}{u_k}\right| = \left|\frac{2^{k+1}\cdot x^{k+1}}{\ln(k+3)} \cdot \frac{\ln(k+2)}{2^k\cdot x^k}\right| = \left|2x\,\frac{\ln(k+2)}{\ln(k+3)}\right|$$

$$\lim_{k\to\infty}\left|2x\,\frac{\ln(k+2)}{\ln(k+3)}\right| = \lim_{k\to\infty}|2x|\cdot\lim_{k\to\infty}\left|\frac{\ln(k+2)}{\ln(k+3)}\right|^{*} = |2x|$$

Series converges if $|2x|<1 \Leftrightarrow -\frac{1}{2} < x < \frac{1}{2}$.

At endpoint $x=\frac{1}{2}$: $\sum_{0}^{\infty}\frac{1}{\ln(k+2)}$

$$\frac{1}{\ln(k+2)} > \frac{1}{k+2} \quad \forall\, k \ge 0 \Rightarrow \sum_{0}^{\infty}\frac{1}{\ln(k+2)} > \sum_{0}^{\infty}\frac{1}{k+2}$$

But $\sum_{0}^{\infty}\frac{1}{k+2}$ is a harmonic series which diverges.

$\therefore$ The given series diverges by the comparison test.

At endpoint $x=-\frac{1}{2}$: $\sum_{0}^{\infty}\frac{(-1)^k}{\ln(k+2)}$ is $\begin{cases} i.)\ \text{strictly alternating} \\ ii.)\ |u_k| > |u_{k+1}|\ \forall\, k \ge 0 \\ iii.)\ \lim_{k\to\infty} u_k = 0 \end{cases}$

$\therefore$ Series converges conditionally at $x=-\frac{1}{2}$, and the interval of convergence is: $-\frac{1}{2} \le x < \frac{1}{2}$.

* By l'Hôpital's Rule:

$$\lim_{k\to\infty}\left|\frac{\ln(k+2)}{\ln(k+3)}\right| = \lim_{k\to\infty}\left|\frac{\frac{1}{k+2}}{\frac{1}{k+3}}\right| = \lim_{k\to\infty}\left|\frac{1+\frac{3}{k}}{1+\frac{2}{k}}\right| = 1$$

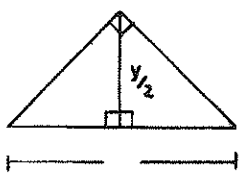
1988-BC5

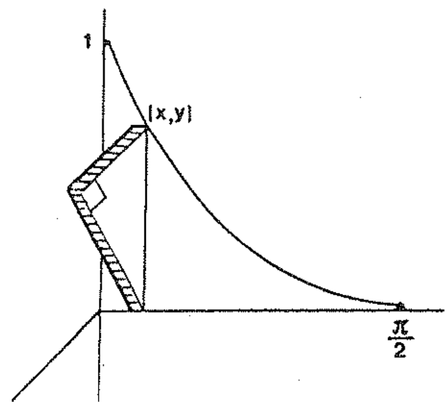
5. The base of a solid S is the shaded region in the xy-plane enclosed by the x-axis, the y-axis, and the graph of $y = 1 - \sin x$, as shown in the figure above. For each x, the cross section of S perpendicular to the x-axis at the point $(x, 0)$ is an isosceles right triangle whose hypotenuse lies in the xy-plane.

(a) Find the area of the triangle as a function of x.
(b) Find the volume of S.

---

a.)





$$A_\Delta = \frac{y^2}{4} = \frac{(1-\sin x)^2}{4}$$

b.) $$V = \frac{1}{4}\int_0^{\pi/2}(1-\sin x)^2\,dx = \frac{1}{4}\int_0^{\pi/2}(1-2\sin x+\sin^2 x)\,dx$$

$$= \frac{1}{4}\int_0^{\pi/2}\left(1-2\sin x+\frac{1}{2}-\frac{\cos 2x}{2}\right)dx$$

$$= \frac{1}{4}\left[\frac{3x}{2}+2\cos x-\frac{\sin 2x}{4}\right]_0^{\pi/2}$$

$$V = \frac{1}{4}\left(\frac{3\pi}{4}-2\right)$$

1988 - BC6

6. Let f be a differentiable function defined for all $x \geqq 0$ such that $f(0) = 5$ and $f(3) = -1$.

Suppose that for any number $b > 0$ the average value of $f(x)$ on the interval $0 \leqq x \leqq b$ is $\dfrac{f(0) + f(b)}{2}$.

(a) Find $\int_0^3 f(x)\,dx$.

(b) Prove that $f'(x) = \dfrac{f(x) - 5}{x}$ for all $x > 0$.

(c) Using part (b), find $f(x)$.

---

a.) $$y_{AV} = \frac{\int_0^3 f(x)dx}{3} = \frac{f(0) + f(3)}{2} = \frac{5-1}{2} = 2$$

$$\int_0^3 f(x)dx = 6$$

b.) $$\frac{\int_0^x f(t)\,dt}{x} = \frac{f(0) + f(x)}{2}, \quad \forall x > 0 \Rightarrow$$

$$\int_0^x f(t)dt = \frac{5x + x f(x)}{2}, \quad \forall x > 0$$

$$f(x) = \tfrac{1}{2}\left(5 + x f'(x) + f(x)\right), \quad \forall x > 0$$

$$f(x) = 5 + x f'(x) \Rightarrow f'(x) = \frac{f(x) - 5}{x}, \quad \forall x > 0$$

c.) From b.): $\frac{dy}{dx} = \frac{y-5}{x} \Rightarrow \frac{dy}{y-5} = \frac{dx}{x}, \quad x > 0$

$$\ln|y-5| = \ln x + C$$

At $(3,-1)$ $\ln 6 = \ln 3 + C \Rightarrow C = \ln 2$

$$\ln|y-5| = \ln x + \ln 2 = \ln 2x$$

$$y - 5 = 2x \quad \text{or} \quad y - 5 = -2x$$

But $(3,-1) \in f$ and does not satisfy $y - 5 = 2x$.

$\therefore \; y - 5 = -2x$ or $y = 5 - 2x$

1989 - AB1

1. Let f be the function given by $f(x) = x^3 - 7x + 6$.

   (a) Find the zeros of f.

   (b) Write an equation of the line tangent to the graph of f at $x = -1$.

   (c) Find the number c that satisfies the conclusion of the Mean Value Theorem for f on the closed interval $[1, 3]$.

---

a.) $f(x) = x^3 - 7x + 6$

$f(1) = 1 - 7 + 6 = 0$

$f(x) = (x-1)(x^2+x-6)$

$= (x-1)(x-2)(x+3)$

$$\begin{array}{cccc|c} 1 & 0 & -7 & 6 & \underline{1} \\ & 1 & 1 & -6 & \\ \hline 1 & 1 & -6 & & \end{array}$$

$\boxed{\text{Zeros: } 1, 2, -3}$

b.) $y - f(-1) = f'(-1)(x+1)$

$f(-1) = -1 + 7 + 6 = 12$

$f'(x) = 3x^2 - 7$

$f'(-1) = 3 - 7 = -4$

$\boxed{y - 12 = -4(x+1)}$

OR

$\boxed{y = -4x + 8}$

c.) $f'(c) = \dfrac{f(3) - f(1)}{3-1}$, $1 < c < 3$

$f(3) = 27 - 21 + 6 = 12$

$f(1) = 0$

$3c^2 - 7 = \frac{12}{2} = 6$

$3c^2 = 13$

$c^2 = \frac{13}{3}$, $1 < c < 3$

$c = \sqrt{\frac{13}{3}}$

1989 - AB 2

2. Let R be the region in the first quadrant enclosed by the graph of $y = \sqrt{6x + 4}$, the line $y = 2x$, and the y-axis.

(a) Find the area of R.

(b) Set up, but do not integrate, an integral expression in terms of a single variable for the volume of the solid generated when R is revolved about the x-axis.

(c) Set up, but do not integrate, an integral expression in terms of a single variable for the volume of the solid generated when R is revolved about the y-axis.

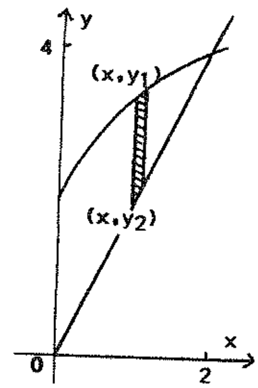
---

a.) $dA = (y_1 - y_2)\,dx$

$$A = \int_0^2 \left(\sqrt{6x+4} - 2x\right) dx$$

$$A = \left.\frac{(6x+4)^{3/2}}{6 \cdot \frac{3}{2}} - x^2 \right]_0^2$$

$$A = \left[\left(\frac{64}{9} - 4\right) - \frac{8}{9}\right] = \frac{20}{9}$$



$$\sqrt{6x+4} = 2x$$
$$6x+4 = 4x^2$$
$$2x^2 - 3x - 2 = 0$$
$$(2x+1)(x-2) = 0$$
$$x = -\tfrac{1}{2},\ 2$$

b.) $dV_x = \pi\left(y_1^2 - y_2^2\right) dx$

$$V_x = \pi \int_0^2 \left(6x + 4 - 4x^2\right) dx$$

c.) $dV_y = 2\pi x\,(y_1 - y_2)\,dx$

$$V_y = 2\pi \int_0^2 x\left(\sqrt{6x+4} - 2x\right) dx$$

1989 - AB 3

3. A particle moves along the x-axis in such a way that its acceleration at time t for $t \geqq 0$ is given by $a(t) = 4\cos(2t)$. At time $t = 0$, the velocity of the particle is $v(0) = 1$ and its position is $x(0) = 0$.

   (a) Write an equation for the velocity $v(t)$ of the particle.

   (b) Write an equation for the position $x(t)$ of the particle.

   (c) For what values of t, $0 \leqq t \leqq \pi$, is the particle at rest?

Given: $a(t) = 4\cos(2t)$, $v(0) = 1$, $x(0) = 0$

a.) $v(t) = \int 4\cos(2t)\,dt = 2\sin(2t) + C_1$

$v(0) = 1 = 2\sin 0 + C_1 \Rightarrow C_1 = 1$

$$\boxed{v(t) = 2\sin(2t) + 1}$$

b.) $x(t) = \int (2\sin(2t) + 1)\,dt$

$= -\cos(2t) + t + C_2$

$x(0) = 0 = -\cos(0) + 0 + C_2 \Rightarrow C_2 = 1$

$$\boxed{x(t) = -\cos(2t) + t + 1}$$

c.) Particle is at rest when $v(t) = 0$

$v(t) = 2\sin(2t) + 1 = 0$ when

$\sin(2t) = -\frac{1}{2}$

Then $2t = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$, or

$$\boxed{t = \frac{7\pi}{12} \text{ or } \frac{11\pi}{12}}$$

1989 – AB 4

4. Let f be the function given by $f(x) = \dfrac{x}{\sqrt{x^2-4}}$.

(a) Find the domain of f.

(b) Write an equation for each vertical asymptote to the graph of f.

(c) Write an equation for each horizontal asymptote to the graph of f.

(d) Find $f'(x)$.

---

a.) D_f: $x < -2$ or $x > 2$

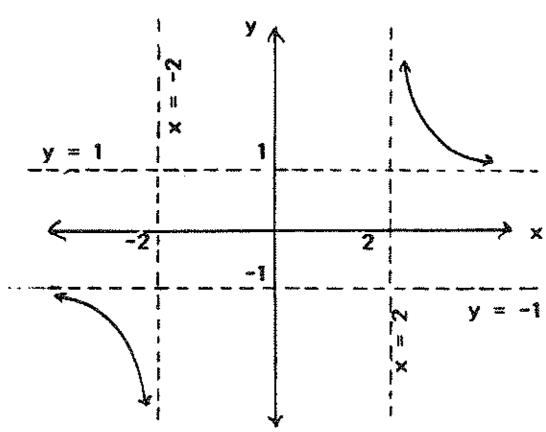
b.) Vertical asymptotes: $x = 2$, $x = -2$

c.) $\displaystyle\lim_{x\to\infty} \frac{x}{\sqrt{x^2-4}} = 1$; $\displaystyle\lim_{x\to-\infty} \frac{x}{\sqrt{x^2-4}} = -1$

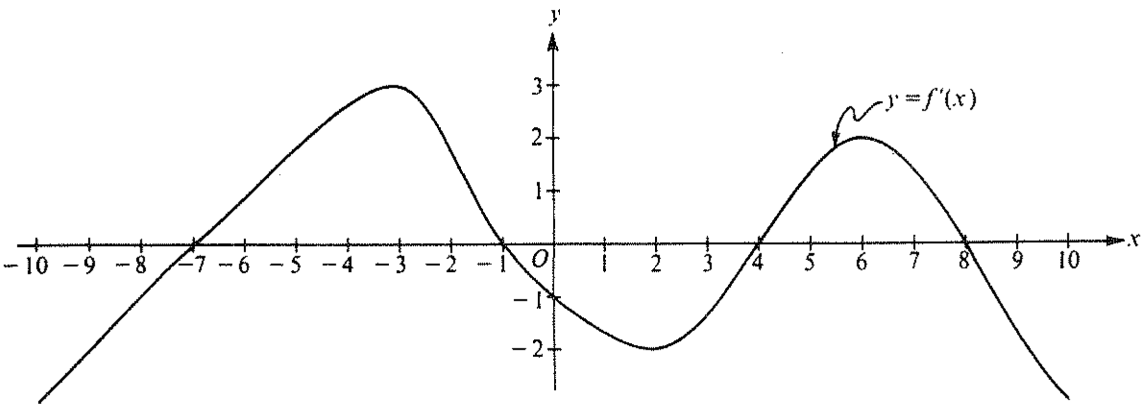
Horizontal asymptotes: $y = 1$, $y = -1$

d.) $$f'(x) = \frac{\sqrt{x^2-4} - \dfrac{x\cdot 2x}{2\sqrt{x^2-4}}}{x^2-4} = \frac{x^2-4-x^2}{(x^2-4)^{3/2}}$$

$$\boxed{f'(x) = \frac{-4}{(x^2-4)^{3/2}}}$$



1989 – AB5



Note: This is the graph of the <u>derivative</u> of f, <u>not</u> the graph of f.

The figure above shows the graph of f', the derivative of a function f. The domain of f is the set of all real numbers x such that $-10 \leqq x \leqq 10$.

(a) For what values of x does the graph of f have a horizontal tangent?

(b) For what values of x in the interval $(-10, 10)$ does f have a relative maximum? Justify your answer.

(c) For what values of x is the graph of f concave downward?

---

a.) f has a horizontal tangent at points where $f'(x) = 0$. This occurs at $x = -7, -1, 4, 8$

b.) $f'(x)$: $-$ on $(-10, -7)$, $+$ on $(-7, -1)$, $-$ on $(-1, 4)$, $+$ on $(4, 8)$, $-$ on $(8, 10)$

f: decr. | incr. | decr. | incr. | decr.

f has a relative max. at $x = -1$ and at $x = 8$

$\left.\begin{array}{l} f \text{ continuous at } x = a \\ f \text{ increasing when } x < a \\ f \text{ decreasing when } x > a \end{array}\right\} \Rightarrow f(a)$ is a relative max.

c.) $f''(x)$: $+$ on $(-10, -3)$, $-$ on $(-3, 2)$, $+$ on $(2, 6)$, $-$ on $(6, 10)$

f is concave down when $-3 < x < 2$ or $6 < x < 10$

1989 – AB6

6. Oil is being pumped continuously from a certain oil well at a rate proportional to the amount of oil left in the well; that is, $\frac{dy}{dt} = ky$, where y is the amount of oil left in the well at any time t. Initially there were 1,000,000 gallons of oil in the well, and 6 years later there were 500,000 gallons remaining. It will no longer be profitable to pump oil when there are fewer than 50,000 gallons remaining.

(a) Write an equation for y, the amount of oil remaining in the well at any time t.

(b) At what rate is the amount of oil in the well decreasing when there are 600,000 gallons of oil remaining?

(c) In order not to lose money, at what time t should oil no longer be pumped from the well?

---

a.) $\frac{dy}{dt} = ky, \quad y(0) = 10^6, \quad y(6) = 5 \times 10^5$

$\frac{dy}{y} = k\,dt \Rightarrow \ln|y| = kt + C_1 \Rightarrow y = Ce^{kt}$

$y(0) = 10^6 = C; \; y(6) = 5 \times 10^5 = 10^6 e^{6k} \Rightarrow e^{6k} = \frac{1}{2}$

$\ln e^{6k} = 6k = \ln\frac{1}{2} \Rightarrow k = \frac{1}{6}\ln\frac{1}{2}$

$\boxed{y(t) = 10^6 e^{\frac{t}{6}\ln\frac{1}{2}} \quad \text{OR} \quad y(t) = 10^6\left(\frac{1}{2}\right)^{\frac{t}{6}}}$

b.) Find $\frac{dy}{dt}$ when $y(t) = 600{,}000$

$\frac{dy}{dt} = ky = \left(\frac{1}{6}\ln\frac{1}{2}\right)(600{,}000) = -100{,}000\ln 2$

The amount of oil is decreasing at rate of $100{,}000 \ln 2$ gallons/year when $y(t) = 600{,}000$.

c.) It is not profitable to pump oil when $y \leq 50{,}000$

$y(t) = 5 \times 10^4 = 10^6 e^{(-\ln 2)t/6} \Rightarrow \frac{1}{20} = e^{-(\ln 2)t/6}$

$-\ln 20 = -\ln 2\left(\frac{t}{6}\right) \Rightarrow t = \frac{6\ln 20}{\ln 2}$

Since y is strictly decreasing, it is not profitable to pump oil when $t \geq 6\ln 20/\ln 2$

1989 - BC 1

1. Let f be a function such that $f''(x) = 6x + 8$.

(a) Find $f(x)$ if the graph of f is tangent to the line $3x - y = 2$ at the point $(0, -2)$.

(b) Find the average value of $f(x)$ on the closed interval $[-1, 1]$.

---

a.) $f'(x) = \int (6x+8)\,dx = 3x^2 + 8x + C_1$

Slope of $3x - y = 2$ is 3. Then $f'(0) = 3 = C_1$

$f'(x) = 3x^2 + 8x + 3$

$f(x) = \int (3x^2 + 8x + 3)\,dx = x^3 + 4x^2 + 3x + C_2$

$(0,-2) \in f \Rightarrow f(0) = -2 = C_2$

$$\boxed{f(x) = x^3 + 4x^2 + 3x - 2}$$

b.) $f_{AV} = \dfrac{\int_{-1}^{1} f(x)\,dx}{2}$

$$= \frac{\left.\frac{x^4}{4} + \frac{4x^3}{3} + \frac{3x^2}{2} - 2x\right]_{-1}^{1}}{2}$$

$$= \frac{\left[\left(\frac{1}{4} + \frac{4}{3} + \frac{3}{2} - 2\right) - \left(\frac{1}{4} - \frac{4}{3} + \frac{3}{2} + 2\right)\right]}{2}$$

$$\boxed{f_{AV} = \frac{2\left(\frac{4}{3} - 2\right)}{2} = \frac{-2}{3}}$$

1989 - BC 2

2. Let R be the region enclosed by the graph of $y = \frac{x^2}{x^2+1}$, the line $x = 1$, and the x-axis.

(a) Find the area of R.
(b) Find the volume of the solid generated when R is rotated about the y-axis.

---

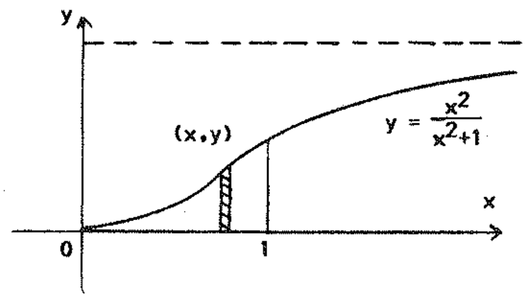
a.) $dA = y\,dx$

$$A = \int_0^1 y\,dx = \int_0^1 \frac{x^2}{x^2+1}\,dx$$

$$A = \int_0^1 \left(1 - \frac{1}{x^2+1}\right)dx$$

$$A = x - \text{Arctan}\, x \Big]_0^1$$

$$\boxed{A = 1 - \frac{\pi}{4}}$$



$$\begin{array}{r} 1 - \frac{1}{x^2+1} \\ x^2+1 \overline{)\, x^2 \qquad} \\ \underline{x^2+1} \\ -1 \end{array}$$

b.) By "shells":

$$dV_y = 2\pi r h\,dx = 2\pi x y\,dx$$

$$V_y = 2\pi \int_0^1 x \cdot \frac{x^2}{x^2+1}\,dx = 2\pi \int_0^1 x\left(1 - \frac{1}{x^2+1}\right)dx$$

$$V_y = 2\pi \int_0^1 \left(x - \frac{x}{x^2+1}\right)dx$$

$$V_y = 2\pi \left[\frac{x^2}{2} - \frac{\ln(x^2+1)}{2}\right]_0^1$$

$$\boxed{V_y = \pi(1 - \ln 2)}$$

1989 – BC 3

3. Consider the function f defined by $f(x) = e^x \cos x$ with domain $[0, 2\pi]$.

(a) Find the absolute maximum and minimum values of $f(x)$.
(b) Find the intervals on which f is increasing.
(c) Find the x-coordinate of each point of inflection of the graph of f.

---

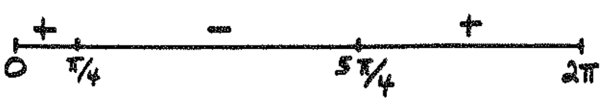
a) $f(x) = e^x \cos x, \quad x \in [0, 2\pi]$
$f'(x) = -e^x \sin x + e^x \cos x = e^x(\cos x - \sin x)$
$f'(x) = 0$ when $\cos x - \sin x = 0$ or $\tan x = 1$
$f'(x) = 0 \Rightarrow x = \pi/4$ or $5\pi/4$

Critical values

x	$f(x)$
0	1
$\pi/4$	$e^{\pi/4}/\sqrt{2}$
$5\pi/4$	$-e^{5\pi/4}/\sqrt{2}$
2π	$e^{2\pi}$

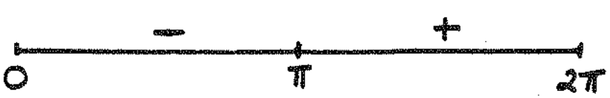
$\boxed{f \text{ has an abs. min. of } \frac{-e^{5\pi/4}}{\sqrt{2}} \text{ and an abs. max. of } e^{2\pi}}$

b) $f'(x) = e^x(\cos x - \sin x)$ (See graph below)

$f'(x)$:


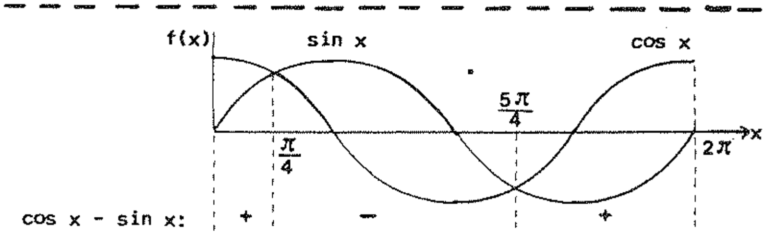
$\boxed{f \text{ is increasing on } [0, \pi/4] \text{ or } [5\pi/4, 2\pi]}$

c) $f''(x) = e^x(-\sin x - \cos x) + e^x(\cos x - \sin x) = -2e^x \sin x$

$f''(x)$:


$\boxed{f \text{ has a point of inflection at } x = \pi}$

Graph for b:



1989 - BC 4

4. Consider the curve given by the parametric equations

$$x = 2t^3 - 3t^2 \quad \text{and} \quad y = t^3 - 12t.$$

(a) In terms of t, find $\frac{dy}{dx}$.

(b) Write an equation for the line tangent to the curve at the point where $t = -1$.

(c) Find the x- and y-coordinates for each critical point on the curve and identify each point as having a vertical or horizontal tangent.

---

a.) $y = t^3 - 12t$ $\qquad$ $x = 2t^3 - 3t^2$

$\frac{dy}{dt} = 3t^2 - 12$ $\qquad$ $\frac{dx}{dt} = 6t^2 - 6t$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 12}{6t^2 - 6t} = \frac{t^2 - 4}{2t^2 - 2t}$$

b.) When $t = -1$, $x = -5$, $y = 11$, $\frac{dy}{dx} = \frac{1-4}{2+2} = \frac{-3}{4}$

Equation of line tangent to curve:

$$\boxed{y - 11 = \frac{-3}{4}(x + 5) \quad \text{OR} \quad 3x + 4y = 29}$$

c.) Critical points occur when $t = 0, 1, 2, -2$.

t	Critical points	Tangent
0	$x = 0$, $y = 0$	Vertical
1	$x = -1$, $y = -11$	Vertical
2	$x = 4$, $y = -16$	Horizontal
-2	$x = -28$, $y = 16$	Horizontal

1989 - BC 5

5. At any time $t \geqq 0$, the velocity of a particle traveling along the x-axis is given by the differential equation $\frac{dx}{dt} - 10x = 60e^{4t}$.

   (a) Find the general solution $x(t)$ for the position of the particle.
   (b) If the position of the particle at time $t = 0$ is $x = -8$, find the particular solution $x(t)$ for the position of the particle.
   (c) Use the particular solution from part(b) to find the time at which the particle is at rest.

---

a.) Integrating Factor:

$\rho = e^{\int -10\,dt} = e^{-10t}$

Multiply both members of given equation by e^{-10t}

$$e^{-10t}\frac{dx}{dt} - 10xe^{-10t} = 60e^{-6t}$$

$$\frac{d(xe^{-10t})}{dt} = 60e^{-6t}$$

$$\int d(xe^{-10t}) = \int 60e^{-6t}\,dt$$

$$xe^{-10t} = -10e^{-6t} + C$$

$$\boxed{x(t) = -10e^{4t} + Ce^{10t}}$$

b.) $x(0) = -8 \Rightarrow -8 = -10 + C$, or $C = 2$

$$\boxed{x(t) = -10e^{4t} + 2e^{10t}}$$

c.) Particle at rest when $\frac{dx}{dt} = 0$

$$\frac{dx}{dt} = -40e^{4t} + 20e^{10t}$$

$$\frac{dx}{dt} = 20e^{4t}(-2 + e^{6t})$$

$\frac{dx}{dt} = 0$ when $-2 + e^{6t} = 0$

or $e^{6t} = 2 \Rightarrow t = \frac{\ln 2}{6}$

a.) Method of Undetermined Coefficients:

i.) Homogeneous Solution (x_H):

$\frac{dx}{dt} - 10x = 0$ or $\frac{dx}{x} = 10\,dt$

$$\ln x = 10t + C$$

$$x_H = e^{10t + C} = C_1e^{10t}$$

ii.) Particular Solution (x_p):

Let $x_p = C_2e^{4t}$

$$\frac{dx_p}{dt} = 4C_2e^{4t}$$

Substituting in the given equation,

$$4C_2e^{4t} - 10C_2e^{4t} = 60e^{4t}$$

$C_2 = -10$ and $x_p = -10e^{4t}$

iii.) General Solution:

$$x_G = x_H + x_p$$

$$\boxed{x_G = C_1e^{10t} - 10e^{4t}}$$

b.) Same

c.) Same

1989 - BC 6

6. Let f be a function that is everywhere differentiable and that has the following properties.

(i) $f(x+h) = \dfrac{f(x)+f(h)}{f(-x)+f(-h)}$ for all real numbers h and x.

(ii) $f(x) > 0$ for all real numbers x.

(iii) $f'(0) = -1$.

(a) Find the value of $f(0)$.
(b) Show that $f(-x) = \dfrac{1}{f(x)}$ for all real numbers x.
(c) Using part (b), show that $f(x+h) = f(x)f(h)$ for all real numbers h and x.
(d) Use the definition of the derivative to find $f'(x)$ in terms of $f(x)$.

---

a.) Substitute $x = h = 0$ in (i):

$$f(0) = \frac{f(0)+f(0)}{f(0)+f(0)} = 1 \quad ; \quad f(x) > 0 \ \forall x \quad \text{(ii)}$$

b.) Let $h = 0$ in (i):

$$f(x) = \frac{f(x)+f(0)}{f(-x)+f(0)} = \frac{f(x)+1}{f(-x)+1} \Rightarrow f(x)\cdot f(-x) + \cancel{f(x)} = \cancel{f(x)} + 1$$

$$\boxed{f(-x) = \frac{1}{f(x)}} \qquad f(x) > 0 \ \forall x \quad \text{(ii)}$$

c.) $$f(x+h) = \frac{f(x)+f(h)}{\frac{1}{f(x)} + \frac{1}{f(h)}} \quad ; \quad \text{(i) and (b)}$$

$$\boxed{f(x+h) = f(x)\cdot f(h)}$$

d.) $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{f(x)\cdot f(h) - f(x)}{h}$$

$$= \lim_{x\to 0} f(x)\left(\frac{f(h)-1}{h}\right) = \lim_{h\to 0} f(x)\cdot \lim_{h\to 0} \frac{f(h)-f(0)}{h}$$

$$= \lim_{h\to 0} f(x)\cdot f'(0) = f(x)(-1) \qquad \text{(iii)}$$

$$\boxed{f'(x) = -f(x)}$$

1990 – AB1

1. A particle, initially at rest, moves along the x-axis so that its acceleration at any time $t \geq 0$ is given by $a(t) = 12t^2 - 4$. The position of the particle when $t = 1$ is $x(1) = 3$.

   (a) Find the values of t for which the particle is at rest.

   (b) Write an expression for the position $x(t)$ of the particle at any time $t \geq 0$.

   (c) Find the total distance traveled by the particle from $t = 0$ to $t = 2$.

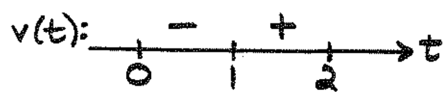
---

a.) $v(t) = \int a(t)\,dt = 4t^3 - 4t + C, \quad t \geq 0$

$v(0) = 0 \Rightarrow C = 0$ and $v(t) = 4t^3 - 4t = 4t(t^2-1), t \geq 0$

$v(t) = 0$ when $t = 0, \pm 1$. But $t \geq 0$

$\boxed{v(t) = 0 \text{ when } t = 0, 1}$



b.) $x(t) = \int v(t)\,dt = t^4 - 2t^2 + C_1$

$x(1) = 3 \Rightarrow 1 - 2 + C_1 = 3$ OR $C_1 = 4$

$\boxed{x(t) = t^4 - 2t^2 + 4, \quad t \geq 0}$

c.) Total distance (TD) traveled from $t=0$ to $t=2$

t	$x(t)$	
0	4	
		>1
1	3	
		>9
2	12	

$\boxed{TD = 10}$

OR

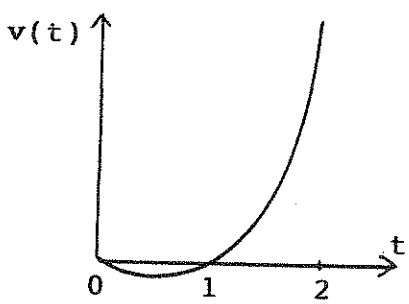
$TD\Big]_0^2 = \left|\int_0^1 (4t^3-4t)\,dt\right| + \left|\int_1^2 (4t^3-4t)\,dt\right|$

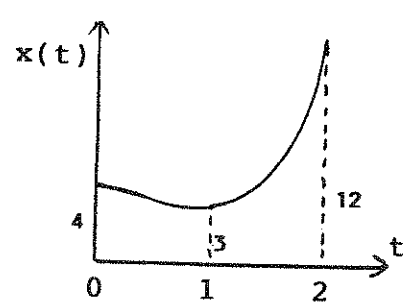
$= \left[2t^2 - t^4\right]_0^1 + \left[t^4 - 2t^2\right]_1^2$

$= 2 - 1 + (8 - (-1)) = 10$

$TD = 10$

Graphs:





1990 – AB 2

2. Let f be the function given by $f(x) = \ln\left(\frac{x}{x-1}\right)$.

(a) What is the domain of f?

(b) Find the value of the derivative of f at $x = -1$.

(c) Write an expression for $f^{-1}(x)$, where f^{-1} denotes the inverse function of f.

---

a.) $\frac{x}{x-1} > 0 \quad \begin{cases} \Rightarrow x > 0 \text{ and } x > 1 \Rightarrow x > 1,\ x \neq 1 \\ \Rightarrow x < 0 \text{ and } x - 1 < 0 \Rightarrow x < 0 \end{cases}$

$$\boxed{D(f) = \{x \mid x > 1 \text{ or } x < 0\}}$$

b.) $\frac{d\left(\ln\left(\frac{x}{x-1}\right)\right)}{dx} = \frac{x-1}{x} \cdot \frac{(x-1)-x}{(x-1)^2} = \frac{-1}{x(x-1)}, \quad x > 1,\ x < 0$

OR $\ln\left(\frac{x}{x-1}\right) = \begin{cases} \ln x - \ln(x-1), & x > 1 \\ \ln(-x) - \ln(1-x), & x < 0 \end{cases}$

$$\frac{d\left(\ln\left(\frac{x}{x-1}\right)\right)}{dx} = \begin{cases} \frac{1}{x} - \frac{1}{x-1} = \frac{-1}{x(x-1)}, & x > 1 \\ \frac{-1}{-x} - \frac{-1}{1-x} = \frac{-1}{x(x-1)}, & x < 0 \end{cases}$$

$$\boxed{f'(-1) = \frac{-1}{2}}$$

c.) $f: \ y = \ln\left(\frac{x}{x-1}\right)$

$f^{-1}: \ x = \ln\left(\frac{y}{y-1}\right)$

$\frac{y}{y-1} = e^x$

$y = \frac{e^x}{e^x - 1}$ (dividendo)

OR $f(f^{-1}(x)) = x$

$\ln\left(\frac{f^{-1}(x)}{f^{-1}(x) - 1}\right) = x$

$\frac{f^{-1}(x)}{f^{-1}(x) - 1} = e^x$

$f^{-1}(x) = \frac{e^x}{e^x - 1}$

1990 – AB 3

3. Let R be the region enclosed by the graphs of $y = e^x$, $y = (x-1)^2$, and the line $x = 1$.

   (a) Find the area of R.

   (b) Find the volume of the solid generated when R is revolved about the $\underline{x\text{-axis}}$.

   (c) Set up, but do not integrate, an integral expression in terms of a single variable for the volume of the solid generated when R is revolved about the $\underline{y\text{-axis}}$.

---

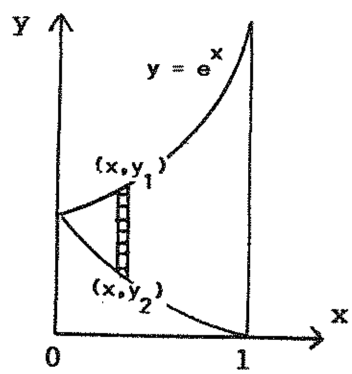
a.) $dA = (y_1 - y_2)\,dx$

$$A = \int_0^1 [e^x - (x-1)^2]\,dx$$

$$= \left[e^x - \frac{(x-1)^3}{3}\right]_0^1$$

$$= \left[(e-1) - \left(0 - \frac{-1}{3}\right)\right]$$

$$\boxed{A = e - \frac{4}{3}}$$



b.) $V_x = \int_0^1 \pi (y_1^2 - y_2^2)\,dx$

$$= \pi \int_0^1 [e^{2x} - (x-1)^4]\,dx$$

$$= \pi \left[\frac{e^{2x}}{2} - \frac{(x-1)^5}{5}\right]_0^1$$

$$= \pi \left[\left(\frac{e^2}{2}\right) - \left(\frac{1}{2} + \frac{1}{5}\right)\right]$$

$$\boxed{V_x = \pi\left(\frac{e^2}{2} - \frac{7}{10}\right)}$$

c.) $V_y = \int_0^1 2\pi r h\,dx = 2\pi \int_0^1 x(y_1 - y_2)\,dx$

$$\boxed{V_y = 2\pi \int_0^1 x\left(e^x - (x-1)^2\right)dx}$$

1990 - AB4

4. The radius r of a sphere is increasing at a constant rate of 0.04 centimeters per second. (Note: The volume of a sphere with radius r is $V = \frac{4}{3}\pi r^3$.)

(a) At the time when the radius of the sphere is 10 centimeters, what is the rate of increase of its volume?

(b) At the time when the volume of the sphere is 36π cubic centimeters, what is the rate of increase of the area of a cross section through the center of the sphere?

(c) At the time when the volume and the radius of the sphere are increasing at the same numerical rate, what is the radius?

---

a.) Find $\frac{dV}{dt}$ when $r = 10$.

$$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = \frac{4\pi}{3} \cdot 3r^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\left.\frac{dV}{dt}\right|_{r=10} = 4\pi(100)(.04) = 16\pi$$

$\boxed{\text{V is changing at the rate of } 16\pi \text{ cm}^3/\text{sec when } r = 10 \text{ cm.}}$

b.) Find $\frac{dA}{dt}$ when $V = 36\pi$.

$$\frac{4}{3}\pi r^3 = 36\pi \Rightarrow r^3 = 27 \text{ and } r = 3$$

$$A = \pi r^2 \Rightarrow \frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 2\pi(3)(.04) = .24\pi$$

$\boxed{\text{A is changing at the rate of } .24\pi \text{ cm}^2/\text{sec when } r = 3 \text{ cm.}}$

c.) Find r when $\frac{dV}{dt}$ and $\frac{dr}{dt}$ are numerically equal.

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \Rightarrow 4\pi r^2 = 1 \text{ and } r = \sqrt{\frac{1}{4\pi}}$$

$\boxed{r = \frac{1}{2\sqrt{\pi}} \text{ cm. when } \frac{dV}{dt} \text{ and } \frac{dr}{dt} \text{ are numerically equal.}}$

1990 – AB5

5. Let f be the function defined by $f(x) = \sin^2 x - \sin x$ for $0 \le x \le \frac{3\pi}{2}$.

(a) Find the x-intercepts of the graph of f.

(b) Find the intervals on which f is increasing.

(c) Find the absolute maximum value and the absolute minimum value of f. Justify your answer.

---

a.) $f(x) = \sin x(\sin x - 1) \qquad 0 \le x \le \frac{3\pi}{2}$

$f(x) = 0$ if $\sin x = 0$ or $\sin x = 1$

$f(x) = 0$ if $x = 0, \pi, \frac{\pi}{2}$

$\boxed{x\text{-intercepts: } 0, \frac{\pi}{2}, \pi}$

b.) f is increasing when $f'(x) > 0$.

$f'(x) = 2\sin x\cos x - \cos x = \cos x(2\sin x - 1)$

$f'(x) = 0$ when $\cos x = 0$ or $\sin x = \frac{1}{2}$

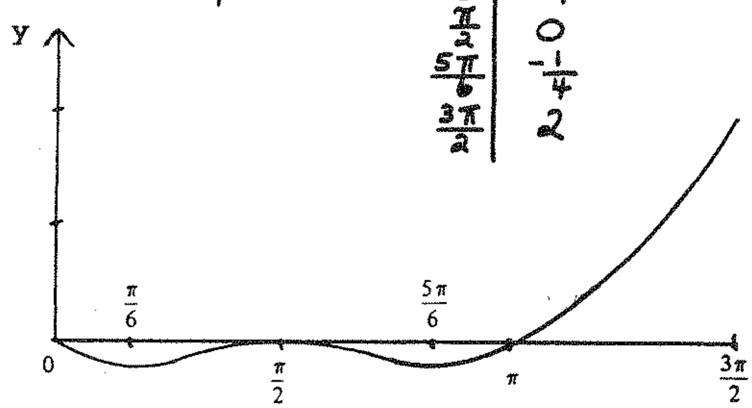
$f'(x) = 0$ when $x = \frac{\pi}{2}$ or $x = \frac{\pi}{6}, \frac{5\pi}{6}$

$f'(x)$	$-$	$+$	$-$	$+$
x: 0	$\frac{\pi}{6}$	$\frac{\pi}{2}$	$\frac{5\pi}{6}$ π	$\frac{3\pi}{2}$

$\boxed{f(x) \text{ is increasing when } \frac{\pi}{6} \le x \le \frac{\pi}{2} \text{ or } \frac{5\pi}{6} \le x \le \frac{3\pi}{2}}$

c.) By the closed interval test, f has an absolute max. of 2 and an absolute min. of $-\frac{1}{4}$.

x	$f(x)$
0	0
$\frac{\pi}{6}$	$-\frac{1}{4}$
$\frac{\pi}{2}$	0
$\frac{5\pi}{6}$	$-\frac{1}{4}$
$\frac{3\pi}{2}$	2



1990 AB5

1990 - AB 6

6. Let f be the function that is given by $f(x) = \dfrac{ax + b}{x^2 - c}$ and that has the following properties.

(i) The graph of f is symmetric with respect to the y-axis.

(ii) $\lim\limits_{x \to 2^+} f(x) = +\infty$

(iii) $f'(1) = -2$

(a) Determine the values of a, b, and c.

(b) Write an equation for each vertical and each horizontal asymptote of the graph of f.

(c) Sketch the graph of f in the xy-plane provided below.

---

a.) y-axis symmetry $\Rightarrow f(x) = f(-x)$

$$\frac{ax+b}{x^2-c} = \frac{-ax+b}{x^2-c} \Rightarrow ax+b = -ax+b \Rightarrow 2ax = 0 \quad \forall x \neq \pm\sqrt{c}$$

$\therefore \boxed{a=0}$ $\quad$ OR f is an even fn $\Rightarrow a = 0$

$$\lim_{x\to 2^+} f(x) = \lim_{x \to 2^+} \frac{b}{x^2-c} = +\infty \Rightarrow \lim_{x\to 2^+}(x^2-c) = 0$$

$$\lim_{x\to 2^+}(4-c) = 0 \Rightarrow \boxed{c=4}\ ; \quad f(x) = \frac{b}{x^2-4}$$

$$f'(x) = -b(x^2-4)^{-2} \cdot 2x$$

$$f'(1) = -b\,(-3)^{-2} \cdot 2 = \frac{-2b}{9} = -2 \ (iii)$$

$$\boxed{b=9}\ ; \quad f(x) = \frac{9}{x^2-4}$$

b.) Vertical asymptotes:

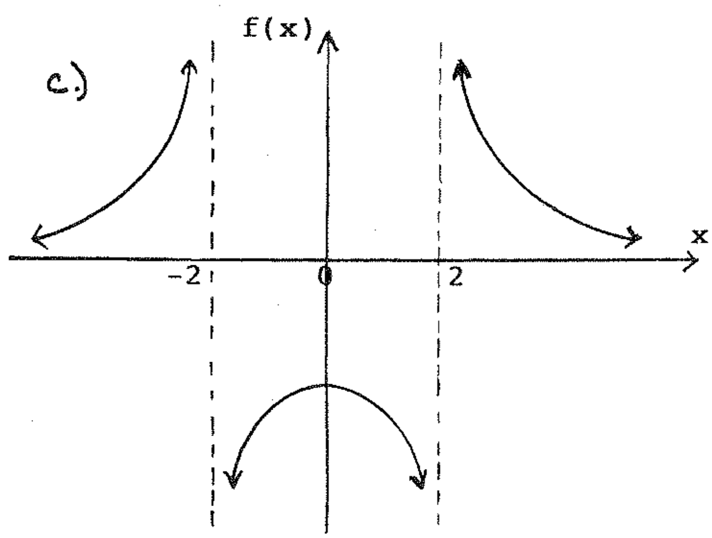
$$\boxed{x = \pm 2}$$

Horizontal asymptotes:

$$\lim_{x\to\pm\infty} \frac{9}{x^2-4} = 0$$

$$\boxed{y=0}$$

c.)



1990 – BC 1

1. A particle starts at time $t = 0$ and moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = (t - 1)^3(2t - 3)$.

   (a) Find the velocity of the particle at any time $t \geq 0$.

   (b) For what values of t is the velocity of the particle less than zero?

   (c) Find the value of t when the particle is moving and the acceleration is zero.

---

a.) $x(t) = (t-1)^3(2t-3), \quad t \geq 0$

$$v(t) = (t-1)^3 \cdot 2 + 3(t-1)^2(2t-3)$$
$$= (t-1)^2\left[2(t-1) + 3(2t-3)\right]$$
$$= (t-1)^2(8t-11)$$

$$\boxed{v(t) = (t-1)^2(8t-11)}$$

b.) $v(t)$: sign chart on the t-axis with marks at 0, 1, $11/8$: $-$ on $(0, 1)$, $-$ on $(1, 11/8)$, $+$ after $11/8$

$v(t) = 0$ when $t = 1$ or $11/8$

$$\boxed{v(t) < 0 \text{ when } 0 \leq t < \tfrac{11}{8} \text{ and } t \neq 1}$$

c.) $a(t) = (t-1)^2 \cdot 8 + 2(t-1)(8t-11)$

$$= (t-1)\left[8(t-1) + 2(8t-11)\right]$$
$$= (t-1)(24t-30)$$

$a(t) = 0$ when $t = 1$ or $5/4$

Since particle is not moving when $t = 1$, $a(t) = 0$ when $t = 5/4$.

$$\boxed{\text{Particle is moving and } a(t) = 0 \text{ when } t = \tfrac{5}{4}}$$

1990 - BC2

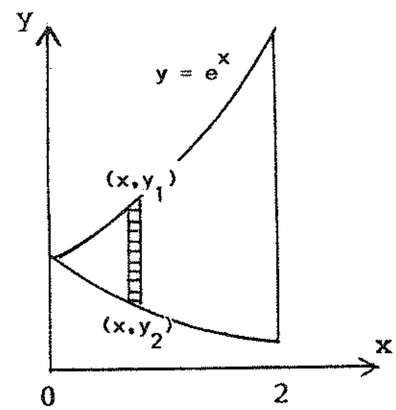
2. Let R be the region in the xy-plane between the graphs of $y = e^{x}$ and $y = e^{-x}$ from $x = 0$ to $x = 2$.

(a) Find the volume of the solid generated when R is revolved about the x-axis.

(b) Find the volume of the solid generated when R is revolved about the y-axis.

---

a.) $dV_x = \pi(y_1^2 - y_2^2)\,dx$



$$V_x = \pi \int_0^2 (e^{2x} - e^{-2x})\,dx$$

$$V_x = \pi \left[\frac{e^{2x}}{2} + \frac{e^{-2x}}{2}\right]_0^2$$

$$= \frac{\pi}{2}\left[(e^4 + e^{-4} - (1+1)\right]$$

$$\boxed{V_x = \frac{\pi}{2}\left[e^4 + e^{-4} - 2\right]}$$

b.) $dV_y = 2\pi r h\,dx$

$$V_y = 2\pi \int_0^2 x(y_1 - y_2)\,dx$$

$$= 2\pi \int_0^2 x(e^{x} - e^{-x})\,dx \qquad \begin{array}{ll} u = x & dv = e^{x}dx \\ du = dx & v = e^{x} \end{array}$$

$$= 2\pi\left[\int_0^2 xe^{x}dx - \int_0^2 xe^{-x}dx\right] \qquad \begin{array}{ll} u = x & dv = e^{-x}dx \\ du = dx & v = -e^{-x} \end{array}$$

$$= 2\pi\left[xe^{x} - e^{x} + xe^{-x} + e^{-x}\right]_0^2$$

$$= 2\pi\left[(2e^2 - e^2 + 2e^{-2} + e^{-2} - (-1+1)\right]$$

$$\boxed{V_y = 2\pi\left(e^2 + 3e^{-2}\right)}$$

1990-BC3

3. Let $f(x) = 12 - x^2$ for $x \geq 0$ and $f(x) \geq 0$.

(a) The line tangent to the graph of f at the point $(k, f(k))$ intercepts the x-axis at $x = 4$. What is the value of k?

(b) An isosceles triangle whose base is the interval from $(0, 0)$ to $(c, 0)$ has its vertex on the graph of f. For what value of c does the triangle have maximum area? Justify your answer.

---

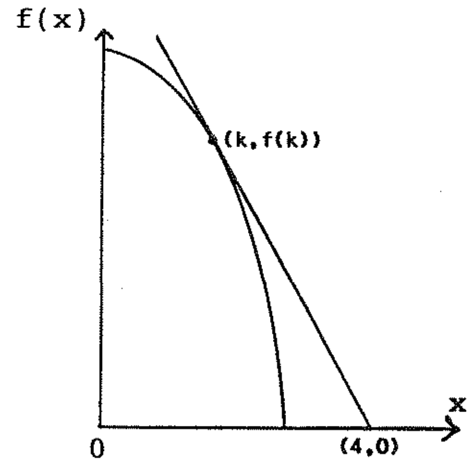
a.) $f'(x) = -2x$

$\dfrac{f(k)-0}{k-4} = -2k$

$x \geq 0$, $f(x) \geq 0$, or

$0 \leq x \leq 2\sqrt{3}$

$0 \leq k \leq 2\sqrt{3}$



$12 - k^2 = -2k^2 + 8k$

$k^2 - 8k + 12 = 0$

$(k-6)(k-2) = 0$

$k = 2$ or 6. But $f(6) < 0$.

$\therefore \boxed{k = 2}$

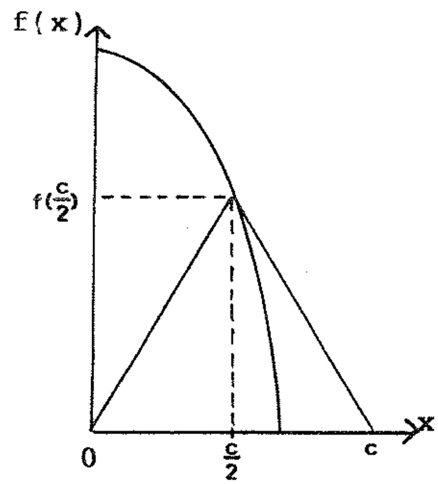
b.) $A = \frac{1}{2} c \left(12 - \frac{c^2}{4}\right)$, $0 < c < 4\sqrt{3}$

$A = 6c - \dfrac{c^3}{8}$

$\dfrac{dA}{dc} = 6 - \dfrac{3c^2}{8} = \dfrac{3(16-c^2)}{8}$

$\dfrac{dA}{dc} = 0$ if $c^2 = 16$ or $c = 4$

$\dfrac{dA}{dc}$: $+$ on $(0, 4)$, $-$ on $(4, 4\sqrt{3})$; marks 0, 4, $4\sqrt{3}$ on the c axis



A has an absolute max. when $c = 4$ by first-derivative test.

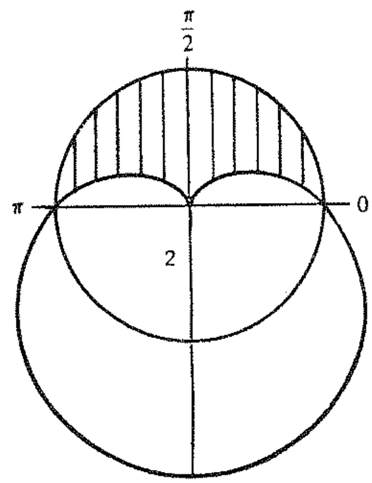
1990 - BC 4

4. Let R be the region inside the graph of the polar curve $r = 2$ and outside the graph of the polar curve $r = 2(1 - \sin\theta)$.

(a) Sketch the two polar curves in the xy-plane provided below and shade the region R.

(b) Find the area of R.

a.)



$$\text{b.) } A = 2\pi - 2\int_0^{\pi/2} \frac{r^2}{2}\,d\theta$$

$$= 2\pi - 2\int_0^{\pi/2} \frac{4(1-\sin\theta)^2}{2}\,d\theta$$

$$= 2\pi - 4\int_0^{\pi/2} (1 - 2\sin\theta + \sin^2\theta)\,d\theta$$

$$= 2\pi - 4\int_0^{\pi/2} \left(1 - 2\sin\theta + \frac{1-\cos 2\theta}{2}\right)d\theta$$

$$= 2\pi - 4\left[\frac{3\theta}{2} + 2\cos\theta - \frac{\sin 2\theta}{4}\right]_0^{\pi/2}$$

$$= 2\pi - 4\left[\frac{3\pi}{4} - 2\right]$$

$$\boxed{A = 8 - \pi}$$

OR $A = \int_0^{\pi} \frac{r_1^2 - r_2^2}{2}\,d\theta$ where $r_1 = 2$ and $r_2 = 2(1 - \sin\theta)$.

1990 - BC 5

5. Let f be the function defined by $f(x) = \dfrac{1}{x-1}$.

(a) Write the first four terms and the general term of the Taylor series expansion of $f(x)$ about $x = 2$.

(b) Use the result from part (a) to find the first four terms and the general term of the series expansion about $x = 2$ for $\ln|x-1|$.

(c) Use the series in part (b) to compute a number that differs from $\ln \frac{3}{2}$ by less than 0.05. Justify your answer.

---

a.) $f(x) = f(a) + f'(a)\dfrac{(x-a)}{1!} + f''(a)\dfrac{(x-a)^2}{2!} + f'''(a)\dfrac{(x-a)^3}{3!} + \dots$

$f(x) = (x-1)^{-1}$	$f'(x) = -1(x-1)^{-2}$	$f''(x) = 2!(x-1)^{-3}$	$f'''(x) = -3!(x-1)^{-4}$
$f(2) = 1$	$f'(2) = -1$	$f''(2) = 2!$	$f'''(2) = -3!$

$f(x) = 1 - (x-2) + (x-2)^2 - (x-2)^3 + \dots + (-1)^n (x-2)^n + \dots$

b.) $\dfrac{d(\ln|x-1|)}{dx} = \dfrac{1}{x-1}$, $x>1 \Rightarrow \ln|x-1| = \int f(x)\,dx$

$\ln|x-1| = x - \dfrac{(x-2)^2}{2} + \dfrac{(x-2)^3}{3} - \dfrac{(x-2)^4}{4} + \dots + (-1)^n \dfrac{(x-2)^{n+1}}{n+1} + \dots + C$

When $x=2$, $\ln 1 = 2 + C \Rightarrow C = -2$

$\ln|x-1| = (x-2) - \dfrac{(x-2)^2}{2} + \dfrac{(x-2)^3}{3} - \dfrac{(x-2)^4}{4} + \dots + (-1)^n \dfrac{(x-2)^{n+1}}{n+1} + \dots$

c.) $x - 1 = \frac{3}{2} \Rightarrow x = \frac{5}{2}$ and $\ln\frac{3}{2} = \frac{1}{2} - \frac{1}{2^2 \cdot 2} + \frac{1}{2^3 \cdot 3} - \frac{1}{2^4 \cdot 4} + \dots$

Since series is strictly alternating and $|u_{n+1}| < |u_n|$, series converges and error $< |u_{n+1}|$

3rd term $= \dfrac{1}{2^3 \cdot 3} = \dfrac{1}{24} < .05$

$\therefore \ln\frac{3}{2} \approx \frac{1}{2} - \frac{1}{8} = .375$

$\boxed{\ln\tfrac{3}{2} \text{ differs from } .375 \text{ by less than } .05}$

1990 – BC 6

6. Let f and g be continuous functions with the following properties.

(i) $g(x) = A - f(x)$ where A is a constant

(ii) $\int_1^2 f(x)dx = \int_2^3 g(x)dx$

(iii) $\int_2^3 f(x)dx = -3A$

(a) Find $\int_1^3 f(x)dx$ in terms of A.

(b) Find the average value of $g(x)$ in terms of A, over the interval [1, 3].

(c) Find the value of k if $\int_0^1 f(x+1)dx = kA$.

---

a.) $\int_1^3 f(x)dx = \int_1^2 f(x)dx + \int_2^3 f(x)dx$

$= \int_2^3 g(x)dx - 3A$

$= \int_2^3 (A - f(x))dx - 3A$

$= \int_2^3 A\,dx - \int_2^3 f(x)dx - 3A$

$= (A + 3A) - 3A$

$\boxed{\int_1^3 f(x)dx = A}$

b.) $g_{avg.}(x) = \dfrac{\int_1^3 g(x)dx}{3-1} = \dfrac{\int_1^3 (A - f(x)dx}{2} = \dfrac{\int_1^3 A\,dx - \int_1^3 f(x)dx}{2}$

$= \frac{1}{2}\left(Ax\Big]_1^3 - A\right) = \frac{A}{2}$

$\boxed{g_{avg.}(x) = \frac{A}{2}}$

c.) $\int_0^1 f(x+1)dx = \int_1^2 f(x)dx = \int_2^3 g(x)dx = kA$

$\int_2^3 g(x)dx = 4A$ (from part a)

$\boxed{k = 4}$

1991 - AB1

1. Let f be the function that is defined for all real numbers x and that has the following properties.

(i) $f''(x) = 24x - 18$ (ii) $f'(1) = -6$ (iii) $f(2) = 0$

(a) Find each x such that the line tangent to the graph of f at $(x, f(x))$ is horizontal.

(b) Write an expression for $f(x)$.

(c) Find the average value of f on the interval $1 \le x \le 3$.

---

a.) Find each x so that $f'(x) = 0$

i.) $f''(x) = 24x - 18 \Rightarrow f'(x) = 12x^2 - 18x + C$

ii.) $f'(1) = -6 = 12 - 18 + C \Rightarrow C = 0$

$f'(x) = 12x^2 - 18x = 6x(2x-3)$

$\boxed{f'(x) = 0 \text{ when } x = 0, 3/2}$

b.) $f(x) = 4x^3 - 9x^2 + C_1$

iii.) $f(2) = 0 \Rightarrow 32 - 36 + C_1 = 0$ OR $C_1 = 4$

$f(x) = \boxed{4x^3 - 9x^2 + 4}$

c.) $\text{Av } f(x)\Big]_1^3 = \dfrac{\int_1^3 (4x^3 - 9x^2 + 4)\,dx}{3-1}$

$= \frac{1}{2}(x^4 - 3x^3 + 4x)\Big]_1^3$

$= \frac{1}{2}\left[(81 - 81 + 12) - (1 - 3 + 4)\right]$

$= \frac{1}{2}(12 - 2) = 5$

$\boxed{\text{Av } f(x)\Big]_1^3 = 5}$

1991 - AB 2

2. Let R be the region between the graphs of $y = 1 + \sin(\pi x)$ and $y = x^2$ from $x = 0$ to $x = 1$.

(a) Find the area of R.

(b) Set up, but do not integrate an integral expression in terms of a single variable for the volume of the solid generated when R is revolved about the x-axis.

(c) Set up, but do not integrate an integral expression in terms of a single variable for the volume of the solid generated when R is revolved about the y-axis.

---

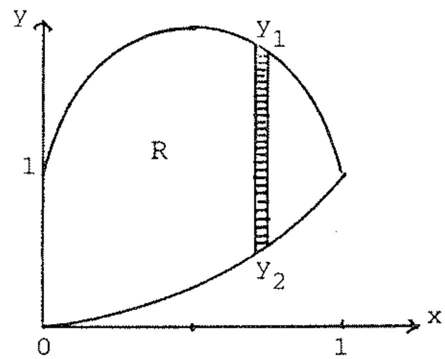
a.) $dA = (y_1 - y_2)dx$

$$A = \int_0^1 (1 + \sin \pi x - x^2)dx$$

$$= x - \frac{\cos \pi x}{\pi} - \frac{x^3}{3} \Big]_0^1$$

$$= \left(1 + \frac{1}{\pi} - \frac{1}{3}\right) - \left(0 - \frac{1}{\pi}\right)$$

$$\boxed{A = \frac{2}{3} + \frac{2}{\pi} \text{ or } 2\left(\frac{1}{3} + \frac{1}{\pi}\right)}$$



b.) "Washers":

$$dV = \pi (R^2 - r^2)\, dx$$

$$V = \int_0^1 \pi (y_1^2 - y_2^2)dx$$

$$\boxed{V = \pi \int_0^1 [(1 + \sin \pi x)^2 - x^4]dx}$$

c.) "Cylindrical Shells":

$$dV = 2\pi r h\, dx$$

$$= 2\pi x (y_1 - y_2)\, dx$$

$$\boxed{V = 2\pi \int_0^1 x(1 + \sin \pi x - x^2)\, dx}$$

1991 – AB3

3. Let f be the function defined by $f(x) = (1 + \tan x)^{\frac{3}{2}}$ for $-\frac{\pi}{4} < x < \frac{\pi}{2}$.

   (a) Write an equation for the line tangent to the graph of f at the point where $x = 0$.

   (b) Using the equation found in part (a), approximate $f(0.02)$.

   (c) Let f^{-1} denote the inverse function of f. Write an expression that gives $f^{-1}(x)$ for all x in the domain of f^{-1}.

---

a.) $y - y_0 = m(x - x_0)$; $m = f'(0)$

$f'(x) = \frac{3}{2}(1 + \tan x)^{1/2} \sec^2 x$

$f'(0) = \frac{3}{2}$; $f(0) = 1$

$$\boxed{y - 1 = \frac{3}{2}x \quad \text{OR} \quad y = \frac{3}{2}x + 1}$$

b.) $\boxed{f(0 + .02) \approx \frac{3}{2}(.02) + 1 = 1.03}$

c.) $y = (1 + \tan x)^{3/2}$; $-\pi/4 < x < \pi/2$, $0 < y < \infty$

f^{-1}: $x = (1 + \tan y)^{3/2}$; $0 < x < \infty$, $-\pi/4 < y < \pi/2$

$\tan y = x^{2/3} - 1$

$$\boxed{f^{-1}(x) = y = \operatorname{Arctan}(x^{2/3} - 1)}$$

1991 − AB 4

4. Let f be the function given by $f(x) = \dfrac{|x| - 2}{x - 2}$.

(a) Find all the zeros of f.

(b) Find $f'(1)$.

(c) Find $f'(-1)$.

(d) Find the range of f.

---

a) $|x| - 2 = 0$ when $x = \pm 2$.
But $f(x)$ is not defined at $x = 2$.

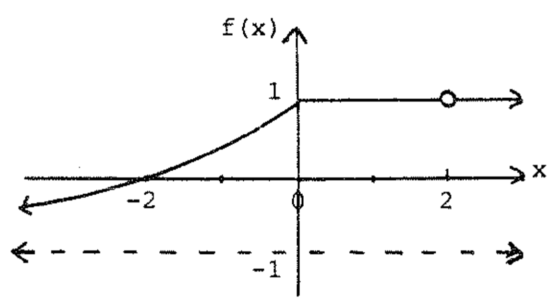
$\boxed{f \text{ has a zero at } x = -2}$

b) $$f(x) = \begin{cases} \dfrac{x-2}{x-2} = 1 & \text{when } x \geq 0,\ x \neq 2 \\ \dfrac{-x-2}{x-2} \text{ OR } \dfrac{x+2}{2-x} \text{ OR } -1 + \dfrac{4}{2-x} & \text{when } x < 0 \end{cases}$$

$$f'(x) = \begin{cases} 0 & \text{when } x > 0,\ x \neq 2 \\ \dfrac{4}{(2-x)^2} & \text{when } x < 0 \end{cases}$$

$\boxed{f'(1) = 0}$

c) $\boxed{f'(-1) = \dfrac{4}{9}}$



d) $\max f(x) = 1$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \left(-1 + \frac{4}{2-x}\right) = -1$$

$\boxed{\text{Range: } -1 < y \leq 1}$

1991 - AB 5

5. Let f be a function that is even and continuous on the closed interval $[-3, 3]$. The function f and its derivatives have the properties indicated in the table below.

x	0	$0 < x < 1$	1	$1 < x < 2$	2	$2 < x < 3$
$f(x)$	1	Positive	0	Negative	-1	Negative
$f'(x)$	Undefined	Negative	0	Negative	Undefined	Positive
$f''(x)$	Undefined	Positive	0	Negative	Undefined	Negative

(a) Find the x-coordinate of each point at which f attains an absolute maximum value or an absolute minimum value. For each x-coordinate you give, state whether f attains an absolute maximum or an absolute minimum.

(b) Find the x-coordinate of each point of inflection on the graph of f. Justify your answer.

(c) In the xy-plane provided below, sketch the graph of a function with all the given characteristics of f.

Note: The xy-plane is provided in the pink test booklet only.

---

a) f is even $\Rightarrow f(x) = f(-x)$, $f'(x) = -f'(x)$; $-1 < f(-3) = f(3) \le 0$

f': − + + − − + x

f: −3 dec −2 inc −1 inc 0 dec 1 dec 2 inc 3

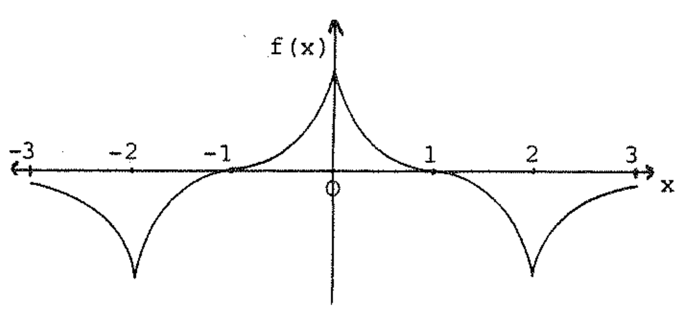
$\boxed{f \text{ has absol. max. at } x = 0\text{; absol. min. at } x = \pm 2}$

b) $f''(x)$: − − + + − − x

−3 −2 −1 0 1 2 3

$\boxed{f \text{ has inflection points at } x = \pm 1}$

c)



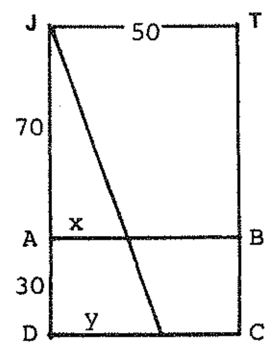
1991– AB6

6. A tightrope is stretched 30 feet above the ground between the Jay and the Tee buildings, which are 50 feet apart. A tightrope walker, walking at a constant rate of 2 feet per second from point A to point B, is illuminated by a spotlight 70 feet above point A, as shown in the diagram.

(a) How fast is the shadow of the tightrope walker's feet moving along the ground when she is midway between the buildings? (Indicate units of measure.)

(b) How far from point A is the tightrope walker when the shadow of her feet reaches the base of the Tee Building? (Indicate units of measure.)

(c) How fast is the shadow of the tightrope walker's feet moving up the wall of the Tee building when she is 10 feet from point B? (Indicate units of measure.)



a.) Given : $dx/dt = 2$

Find : dy/dt when $x = 25$

$\frac{y}{100} = \frac{x}{70} \Rightarrow \frac{dy}{dt} = \frac{100}{70}\frac{dx}{dt}$; $\frac{dy}{dt} = \frac{20}{7}$

$\boxed{\text{Shadow moves along ground at the constant rate of } \frac{20}{7} \text{ ft/sec}}$

b.) Find x when $y = 50$

$\frac{x}{50} = \frac{70}{100} \Rightarrow x = \frac{70 \times 50}{100} = 35$

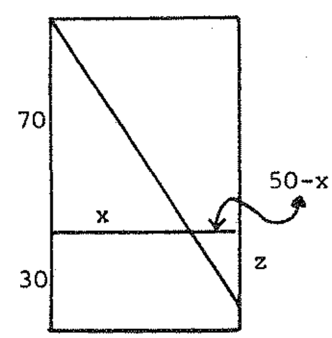
$\boxed{\text{The tightrope walker is 35 ft. from A when the shadow is at C.}}$

c.) Find dz/dt when $x = 40$

$\frac{z}{70} = \frac{50-x}{x} = \frac{50}{x} - 1$

$dz/dt = 70\left(\frac{-50}{x^2}\right) dx/dt$

$x = 40 \Rightarrow dz/dt = \frac{-70 \times 50}{1600} \cdot 2 = \frac{-35}{8}$



$\boxed{\text{When walker is 10' from B, the shadow is moving up the wall at } \frac{35}{8} \text{ ft/sec.}}$

1991 - BC1

1. A particle moves on the x-axis so that its velocity at any time $t \ge 0$ is given by $v(t) = 12t^2 - 36t + 15$. At $t = 1$, the particle is at the origin.

   (a) Find the position $x(t)$ of the particle at any time $t \ge 0$.

   (b) Find all values of t for which the particle is at rest.

   (c) Find the maximum velocity of the particle for $0 \le t \le 2$.

   (d) Find the total distance traveled by the particle from $t = 0$ to $t = 2$.

---

a.) $x(t) = \int v\,dt = 4t^3 - 18t^2 + 15t + C$

$x(1) = 0 = 4 - 18 + 15 + C \Rightarrow C = -1$

$\boxed{x(t) = 4t^3 - 18t^2 + 15t - 1}$

b.) Particle is at rest when $v(t) = 0$

$v(t) = 3(4t^2 - 12t + 5) = 3(2t-1)(2t-5)$

$\boxed{v(t) = 0 \text{ when } t = \tfrac{1}{2}, \tfrac{5}{2}}$

c.) $v'(t) = 3(8t - 12) \Rightarrow v'(t) = 0$ when $t = \frac{3}{2}$, $0 \le t \le 2$

$v'(t)$: on $[0, \frac{3}{2}]$: $-$; on $[\frac{3}{2}, 2]$: $+$; t

$v(\frac{3}{2})$ is an absol. min. $\Rightarrow$ $v(0)$ or $v(2)$ is an absol. max.

$v(0) = 15$; $v(2) = 3(16 - 24 + 5) = -9$

$\boxed{v \text{ has an absol. max. of } 15 \text{ at } t = 0}$

d.)

t	$x(t)$	
0	-1	
		3.5
$\frac{1}{2}$	$\frac{5}{2}$	
		13.5
2	-11	

$\Rightarrow$ $\boxed{\text{Total dist.} = 17}$

OR Total dist. $= \left| \int_0^{1/2} (12t^2 - 36t + 15)\,dt \right| + \left| \int_{1/2}^{2} (12t^2 - 36t + 15)\,dt \right|$

1991 – BC2

2. Let f be the function defined by $f(x) = xe^{1-x}$ for all real numbers x.

(a) Find each interval on which f is increasing.

(b) Find the range of f.

(c) Find the x-coordinate of each point of inflection of the graph of f.

(d) Using the results found in parts (a), (b), and (c), sketch the graph of f in the xy-plane provided below. (Indicate all intercepts.)

---

a.) $f'(x) = xe^{1-x}(-1) + e^{1-x}$

$= e^{1-x}(-x+1)$

$f'(x)$: $+$ | $-$ at 1, x

$\boxed{f \text{ increases when } x \le 1}$

b.) Range of f:

f has an absol. max. at $x=1 \Rightarrow f(x) \le f(1) = 1$

$\lim_{x \to -\infty} xe^{1-x} = -\infty$

$\boxed{R_f : (-\infty, 1] \text{ OR } f(x) \le 1}$

c.) Pts. of inflection

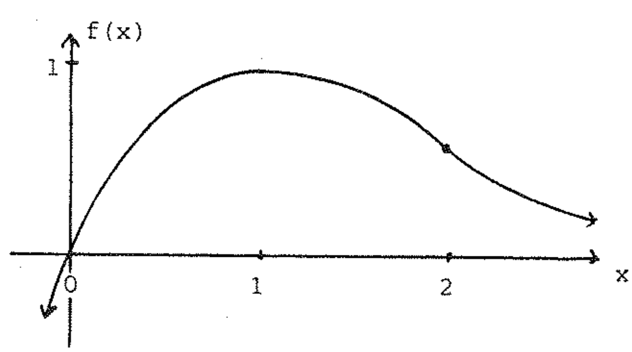
$f''(x) = e^{1-x}(-1) + (1-x)e^{1-x}(-1)$

$= e^{1-x}(-1-1+x) = e^{1-x}(x-2)$

$f''(x)$: $-$ | $+$ at 2, x

$\boxed{f \text{ has a pt. of inflection at } x = 2}$

d.)



1991 – BC 3

3. Let R be the shaded region in the first quadrant enclosed by the y-axis and the graphs of $y = \sin x$ and $y = \cos x$, as shown in the figure above.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is revolved about the x-axis.

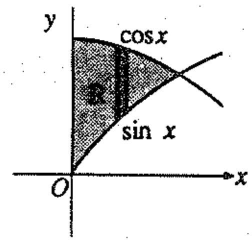
(c) Find the volume of the solid whose base is R and whose cross sections cut by planes perpendicular to the x-axis are squares.

---

a.) $A = \int_0^{\pi/4} (\cos x - \sin x)\,dx$

$= \sin x + \cos x \Big]_0^{\pi/4}$

$\boxed{A = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 1 = \sqrt{2} - 1}$



b.) $dV_x = \pi\,(y_1^2 - y_2^2)\,dx$

$= \pi\,(\cos^2 x - \sin^2 x)\,dx$

$V_x = \pi \int_0^{\pi/4} \cos 2x\,dx$

$\boxed{V_x = \frac{\pi}{2} \sin 2x \Big]_0^{\pi/4} = \frac{\pi}{2}}$

Pt. of intersection:

$\sin x = \cos x \Rightarrow$

$\tan x = 1 \Rightarrow$

$x = \pi/4$

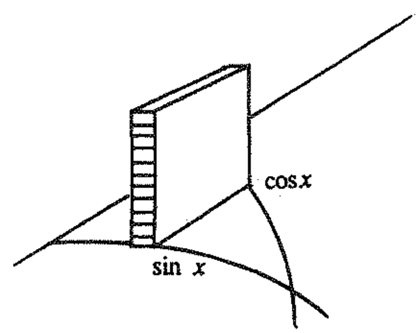
c.) $dV = (\cos x - \sin x)^2\,dx$

$V = \int_0^{\pi/4} (\cos^2 x + \sin^2 x - 2\sin x \cos x)\,dx$

$= \int_0^{\pi/4} (1 - \sin 2x)\,dx$

$= x + \frac{\cos 2x}{2} \Big]_0^{\pi/4}$

$\boxed{V = \frac{\pi}{4} - \frac{1}{2} \text{ OR } \frac{\pi - 2}{4}}$



1991 – BC 4

4. Let $F(x) = \int_1^{2x} \sqrt{t^2 + t}\, dt$.

(a) Find $F'(x)$.

(b) Find the domain of F.

(c) Find $\lim_{x \to \frac{1}{2}} F(x)$.

(d) Find the length of the curve $y = F(x)$ for $1 \le x \le 2$.

---

a.) Let $u = 2x$. Then $F(x) = \int_1^u \sqrt{t^2+t}\, dt$

$$\frac{d(F(x))}{du} = \sqrt{u^2+u} \Rightarrow F'(x) = \frac{d(F(x))}{du} \cdot \frac{du}{dx} = \sqrt{u^2+u}\ \frac{du}{dx}$$

$$\boxed{F'(x) = 2\sqrt{4x^2+2x}}$$

b.) $t^2+t \ge 0$ $\qquad$ t^2+t: $+$ at $t<-1$, $-$ between -1 and 0, $+$ at $t>0$

$f(t) = \sqrt{t^2+t}$ must be cont. on some interval I. $f(t)$ does not exist when $t \in (-1, 0)$; $f(t)$ is cont. if $t \ge 0$ or $t \le -1$. Therefore, $2x \ge 0 \Rightarrow x \ge 0$ and $2x \le -1 \Rightarrow x \le -\frac{1}{2}$. Candidates for I are $[0, \infty)$ and $(-\infty, -\frac{1}{2}]$. Since the lower bound of the integral is 1 and since $1 \in [0, \infty)$, x cannot be negative.

$$\boxed{D_F:\ x \ge 0 \text{ OR } [0, \infty)}$$

c.) Let $G(t) = \int \sqrt{t^2+t}\, dt$. Then $\int_1^{2x} \sqrt{t^2+t}\, dt = G(2x) - G(1)$

$$\lim_{x\to\frac{1}{2}} F(x) = \lim_{x\to\frac{1}{2}} (G(2x) - G(1)) = \lim_{x\to\frac{1}{2}} G(2x) - G(1) = G(1) - G(1).$$

$$\boxed{\lim_{x\to\frac{1}{2}} F(x) = 0}$$

d.) $\ell = \int_1^2 \sqrt{1+(F'(x))^2}\, dx = \int_1^2 \sqrt{1+4(4x^2+2x)}\, dx = \int_1^2 \sqrt{16x^2+8x+1}\, dx$

$$= \int_1^2 (4x+1)\, dx = 2x^2+x\Big]_1^2 = 8+2-3$$

$$\boxed{\ell = 7}$$

1991- BC5

5. Let f be the function given by $f(t) = \dfrac{4}{1+t^2}$ and G be the function given by $G(x) = \int_0^x f(t)dt$.

(a) Find the first four nonzero terms and the general term for the power series expansion of $f(t)$ about $t = 0$.

(b) Find the first four nonzero terms and the general term for the power series expansion of $G(x)$ about $x = 0$.

(c) Find the interval of convergence of the power series in part (b). (Your solution must include an analysis that justifies your answer.)

---

a.) $f(t) = \dfrac{4}{1+t^2} = 1+t^2 \overline{\smash{)}\,4\,}$ with quotient $4 - 4t^2 + 4t^4 - 4t^6 + \ldots + 4(-1)^k t^{2k} + \ldots$

OR $f(t) = 4\sum_{k=0}^{\infty} (-1)^k t^{2k}$

b.) $G(x) = 4t - \dfrac{4t^3}{3} + \dfrac{4t^5}{5} - \dfrac{4t^7}{7} + \ldots + \dfrac{4(-1)^k t^{2k+1}}{2k+1} + \ldots \Big]_0^x$

$G(x) = 4x - \dfrac{4x^3}{3} + \dfrac{4x^5}{5} - \dfrac{4x^7}{7} + \ldots + \dfrac{4(-1)^k x^{2k+1}}{2k+1} + \ldots$

OR $G(x) = 4\sum_{k=0}^{\infty} \dfrac{(-1)^k x^{2k+1}}{2k+1}$

c.) $\lim_{k\to\infty} \left|\dfrac{u_{k+1}}{u_k}\right| = \lim_{k\to\infty} \left|\dfrac{x^{2k+3}}{2k+3} \cdot \dfrac{2k+1}{x^{2k+1}}\right| = \lim_{k\to\infty} \left|\dfrac{2k+1}{2k+3} \cdot x^2\right| = x^2$

Series converges if $x^2 < 1$ OR $-1 < x < 1$

Endpoints:

$x = 1 \Rightarrow G(1) = 4\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right)$

$x = -1 \Rightarrow G(-1) = 4\left(-1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \ldots\right)$

*$G(1)$ and $G(-1)$ converge conditionally by alt. series test.

$\boxed{\text{Interval of Convergence: } -1 \le x \le 1}$

* $G(1)$ and $G(-1)$ converge to π and $-\pi$ respectively

1991 – BC 6

6. A certain rumor spreads through a community at the rate $\frac{dy}{dt} = 2y(1-y)$, where y is the proportion of the population that has heard the rumor at time t.

(a) What proportion of the population has heard the rumor when it is spreading the fastest?

(b) If at time $t = 0$ ten percent of the people have heard the rumor, find y as a function of t.

(c) At what time t is the rumor spreading the fastest?

---

a.) $\frac{d\left(\frac{dy}{dt}\right)}{dy} = 2 - 4y, \quad 0 < y < 1$

$\frac{d\left(\frac{dy}{dt}\right)}{dy}$:

y	0		$\frac{1}{2}$		1
sign		$+$		$-$	

$\frac{dy}{dt}$ is an absol. max. when $y = \frac{1}{2}$

$\boxed{\text{50\% of the pop. has heard the rumor when it is spreading the fastest.}}$

b.) $\frac{dy}{y(1-y)} = 2\,dt \Rightarrow \int\left(\frac{1}{y} + \frac{1}{1-y}\right)dy = 2t + C$

$\ln y - \ln(1-y) = 2t + C \Rightarrow \ln\frac{y}{1-y} = 2t + C$

$\frac{y}{1-y} = e^{2t+C} = e^{2t}\cdot e^{C} \Rightarrow \frac{.1}{1-.1} = \frac{1}{9} = e^{C}$

$\frac{y}{1-y} = \frac{e^{2t}}{9}$ when $y = .1$ and $t = 0$

By componendo:

$\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a}{a+b} = \frac{c}{c+d}$

$$\boxed{y = \frac{e^{2t}}{e^{2t}+9}}$$

c.) $\frac{dy}{dt}$ is absol. max. when $y = \frac{1}{2} \Rightarrow \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{e^{2t}}{9}$

or $e^{2t} = 9 \Rightarrow 2t = \ln 9 = 2\ln 3 \Rightarrow t = \ln 3$

$\boxed{\text{When } t = \ln 3 \text{, the rumor is spreading the fastest.}}$

1992 - AB1

1. Let f be the function defined by $f(x) = 3x^5 - 5x^3 + 2$.

   (a) On what intervals is f increasing?

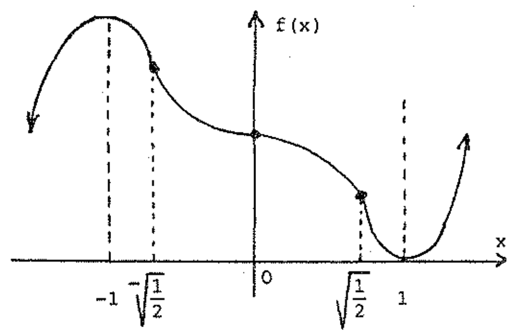
   (b) On what intervals is the graph of f concave upward?

   (c) Write the equation of each horizontal tangent line to the graph of f.

---

a.) $f'(x) = 15x^4 - 15x^2 = 15x^2(x^2 - 1)$

$f'(x)$: $+ \mid - \mid - \mid +$ on x, with divisions at -1, 0, 1



$\boxed{f \text{ is increasing when } x \le -1 \text{ or } x \ge 1}$

b.) $f''(x) = 60x^3 - 30x = 30x(2x^2 - 1)$

$f''(x) = 0$ when $x = 0,\ \pm\sqrt{\frac{1}{2}}$

$f''(x)$: $\begin{cases} 30x: & - \mid - \mid + \mid + \\ 2x^2 - 1: & + \mid - \mid - \mid + \end{cases}$ on x, with divisions at $-\sqrt{\frac{1}{2}}$, 0, $\sqrt{\frac{1}{2}}$

$f''(x)$: $- \mid + \mid - \mid +$

$\boxed{f \text{ is concave up when } -\sqrt{\tfrac{1}{2}} < x < 0 \text{ or } x > \sqrt{\tfrac{1}{2}}}$

c.) $f'(x) = 0$ when $x = 0, \pm 1$

$f(0) = 2 \qquad f(1) = 0 \qquad f(-1) = 4$

$\boxed{\text{Horizontal tangents: } y = 2,\ y = 0,\ y = 4}$

1992 - AB2

2. A particle moves along the x-axis so that its velocity at time t, $0 \le t \le 5$, is given by $v(t) = 3(t-1)(t-3)$. At time $t = 2$, the position of the particle is $x(2) = 0$.

(a) Find the minimum acceleration of the particle.

(b) Find the total distance traveled by the particle.

(c) Find the average velocity of the particle over the interval $0 \le t \le 5$.

---

a.) $v(t) = 3(t-1)(t-3) = 3t^2 - 12t + 9, \quad 0 \le t \le 5$

$a(t) = v'(t) = 3(2t-4) \Rightarrow a(t)$ is increasing $\forall t \in D_f$

Therefore, $a_{min.}$ occurs at left-hand endpoint $t = 0$.

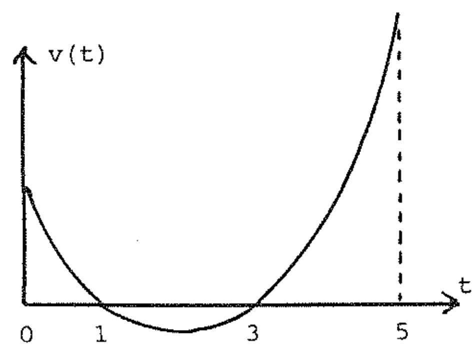
$\boxed{a_{min} = a(0) = -12}$

b.) $x(t) = \int v\,dt = \int 3(t^2 - 4t + 3)\,dt$

$= t^3 - 6t^2 + 9t + C$

$x(2) = 0 = 8 - 24 + 18 + C = 2 + C \Rightarrow C = -2$

$x(t) = t^3 - 6t^2 + 9t - 2$



Distance traveled:

t	x(t)	Dist.
0	-2	
		4
1	2	
		4
3	-2	
		20
5	18	

$\boxed{\text{Total distance traveled} = 28 \text{ units}}$

c.) $v_{av.} = \dfrac{\int_0^5 v(t)\,dt}{5-0} = \left. \dfrac{t^3 - 6t^2 + 9t}{5} \right]_0^5$

$= \dfrac{125 - 150 + 45}{5} = 4$

$\boxed{\text{The average velocity on } [0,5] \text{ is } 4}$

1992 - AB3

3. Let f be the function given by $f(x) = \ln\left|\dfrac{x}{1+x^2}\right|$.

   (a) Find the domain of f.

   (b) Determine whether f is an even function, an odd function, or neither. Justify your conclusion.

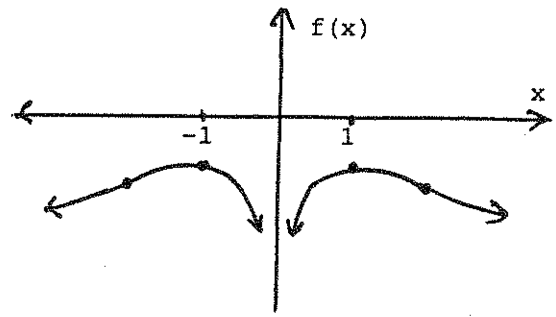
   (c) At what values of x does f have a relative maximum or a relative minimum? For each such x, use the first derivative test to determine whether $f(x)$ is a relative maximum or a relative minimum.

   (d) Find the range of f.

---

a.) D_f = Set of all real $x \neq 0$

b.) f is even if $f(-x) = f(x)$



$$f(-x) = \ln\left|\frac{-x}{1+x^2}\right| = \ln\left|\frac{x}{1+x^2}\right| = f(x)$$

$\boxed{f \text{ is an even function}}$

c.) $x > 0$: $f(x) = \ln x - \ln(1+x^2)$

$$f'(x) = \frac{1}{x} - \frac{2x}{1+x^2} = \frac{1-x^2}{x(1+x^2)}$$

$f'(x) = 0$ if $x = 1$

$f'(x)$: 0 + 1 − → x

$\boxed{f \text{ has a relative max. at } x = 1}$

$x < 0$: $f(x) = \ln(-x) - \ln(1+(-x)^2)$

$$f'(x) = \frac{-1}{-x} - \frac{2x}{1+x^2} = \frac{1-x^2}{x(1+x^2)}$$

$f'(x)$: + −1 − 0 → x

$\boxed{f \text{ has a relative max. at } x = -1}$

d.) $\boxed{R_f:\ f(x) \le \ln\frac{1}{2} = -\ln 2}$

1992-AB4, BC1

4. Consider the curve defined by the equation $y + \cos y = x + 1$ for $0 \le y \le 2\pi$.

(a) Find $\frac{dy}{dx}$ in terms of y.

(b) Write an equation for each vertical tangent to the curve.

(c) Find $\frac{d^2y}{dx^2}$ in terms of y.

---

a.) $\frac{dy}{dx} - \sin y \frac{dy}{dx} = 1$

$\boxed{\frac{dy}{dx} = \frac{1}{1-\sin y}}$ $0 \le y \le 2\pi$ and $y \ne \frac{\pi}{2}$

b.) When $y = \frac{\pi}{2}$, $\frac{dy}{dx}$ is not defined.

$y = \frac{\pi}{2} \Rightarrow \frac{\pi}{2} + 0 = x + 1 \Rightarrow x = \frac{\pi}{2} - 1$

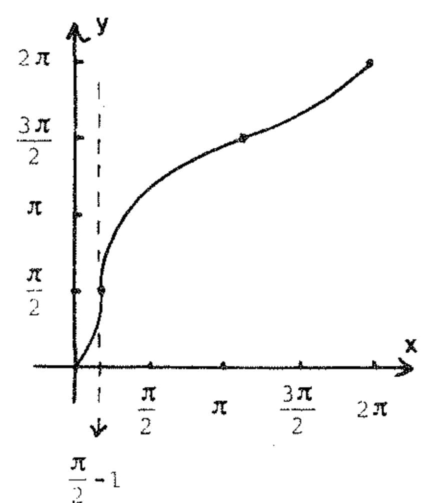
$\boxed{\text{Equation of vertical tangent: } x = \frac{\pi}{2} - 1}$

c.) $\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{d\left(\frac{1}{1-\sin y}\right)}{dx} = \frac{\cos y}{(1-\sin y)^2} \frac{dy}{dx}$

$\boxed{\frac{d^2y}{dx^2} = \frac{\cos y}{(1-\sin y)^3}}$

Graph

y	x
0	0
$\pi/2$	$\pi/2 - 1$
π	$\pi - 2$
$3\pi/2$	$3\pi/2 - 1$
2π	2π



1992 - AB5, BC2

5. Let f be the function given by $f(x) = e^{-x}$, and let g be the function given by $g(x) = kx$, where k is the nonzero constant such that the graph of f is tangent to the graph of g.

(a) Find the x-coordinate of the point of tangency and the value of k.

(b) Let R be the region enclosed by the y-axis and the graphs of f and g. Using the results found in part (a), determine the area of R.

(c) Set up, but do not integrate, an integral expression in terms of a single variable for the volume of the solid generated by revolving the region R, given in part (b), about the x-axis.

---

a.) At point of tangency:

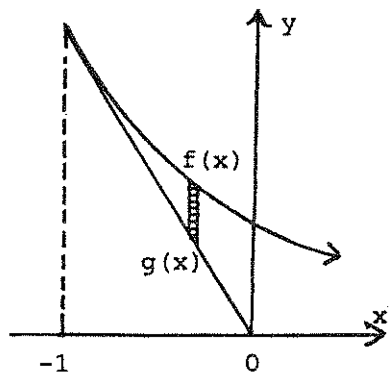
i.) $f(x) = g(x)$ and $f'(x) = g'(x)$

ii.) $e^{-x} = kx$ and $-e^{-x} = k$

$$e^{-x} = -k \Rightarrow -k = kx \Rightarrow x = -1,\ k = -e$$

$\boxed{x\text{- coord. of pt. of tangency} = -1;\ k = -e}$

b.) $dR = [f(x) - g(x)]\,dx$



$$R = \int_{-1}^{0} [e^{-x} - (-ex)]\,dx$$

$$= -e^{-x} + e\frac{x^2}{2}\Big]_{-1}^{0}$$

$$= -1 - (-e + \frac{e}{2})$$

$$\boxed{R = \frac{e}{2} - 1}$$

c.) $dV = \pi (R^2 - r^2)\,dx$

$$\boxed{V = \pi \int_{-1}^{0} [e^{-2x} - (ex)^2]\,dx}$$

1992 – AB 6

6. At time t, $t \geq 0$, the volume of a sphere is increasing at a rate proportional to the reciprocal of its radius. At $t = 0$, the radius of the sphere is 1 and at $t = 15$, the radius is 2. (The volume V of a sphere with radius r is $V = \frac{4}{3}\pi r^3$.)

(a) Find the radius of the sphere as a function of t.

(b) At what time t will the volume of the sphere be 27 times its volume at $t = 0$?

---

a.) Given: $\frac{dV}{dt} = \frac{k}{r}$; when $t=0, r=1$; when $t=15, r=2$

$$\frac{dV}{dt} = \frac{4\pi}{3} 3r^2 \frac{dr}{dt} = \frac{k}{r}$$

$$\int k\,dt = \int 4\pi r^3 dr$$

$$kt + C = \pi r^4$$

$t=0, r=1 \Rightarrow C = \pi$

$t=15, r=2 \Rightarrow 15k + \pi = 16\pi \Rightarrow k = \pi$

$\pi r^4 = \pi t + \pi \Rightarrow r^4 = t+1$

$$\boxed{r = (1+t)^{1/4}}$$

b.) When does $V(t) = 27\,V(0)$?

$V(t) = \frac{4}{3}\pi(1+t)^{3/4}$; $V(0) = \frac{4}{3}\pi$; $27\,V(0) = 36\pi$

$\frac{4}{3}\pi(1+t)^{3/4} = 36\pi \Rightarrow (1+t)^{3/4} = 27$

$1+t = 27^{4/3} = 81$

$$\boxed{V(t) = 27\,V(0) \text{ when } t = 80}$$

1992 - BC 3

3. At time t, $0 \le t \le 2\pi$, the position of a particle moving along a path in the xy-plane is given by the parametric equations $x = e^t \sin t$ and $y = e^t \cos t$.

(a) Find the slope of the path of the particle at time $t = \frac{\pi}{2}$.

(b) Find the speed of the particle when $t = 1$.

(c) Find the distance traveled by the particle along the path from $t = 0$ to $t = 1$.

---

a.) $\frac{dy}{dt} = e^t(-\sin t) + e^t \cos t = e^t(\cos t - \sin t)$

$\frac{dx}{dt} = e^t \cos t + e^t \sin t = e^t(\cos t + \sin t)$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos t - \sin t}{\cos t + \sin t}$$

$$\boxed{t = \frac{\pi}{2} \Rightarrow \frac{dy}{dx} = \frac{\cos\frac{\pi}{2} - \sin\frac{\pi}{2}}{\cos\frac{\pi}{2} + \sin\frac{\pi}{2}} = -1}$$

b.) $v = \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{e^{2t}(\cos t + \sin t)^2 + e^{2t}(\cos t - \sin t)^2}$

$= e^t\sqrt{\cos^2 t + 2\cos t \sin t + \sin^2 t + \cos^2 t - 2\cos t \sin t + \sin^2 t}$

$= e^t\sqrt{2}$

$\boxed{\text{When } t=1, \text{ speed is } e\sqrt{2}}$

c.) $s = \int_{t=0}^{1} v\, dt = \int_0^1 e^t\sqrt{2}\, dt = e^t\sqrt{2}\Big]_0^1$

$= e\sqrt{2} - \sqrt{2} = (e-1)\sqrt{2}$

$\boxed{\text{Distance traveled from } t=0 \text{ to } t=1 \text{ is } \sqrt{2}(e-1)}$

1992 - BC 4

4. Let f be a function defined by $f(x) = \begin{cases} 2x - x^2 & \text{for } x \le 1, \\ x^2 + kx + p & \text{for } x > 1. \end{cases}$

(a) For what values of k and p will f be continuous and differentiable at $x = 1$?

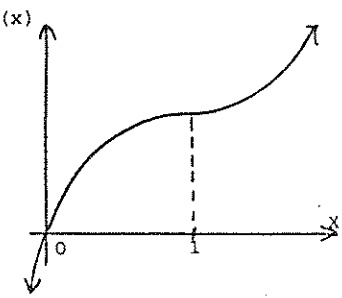
(b) For the values of k and p found in part (a), on what interval or intervals is f increasing?

(c) Using the values of k and p found in part (a), find all points of inflection of the graph of f. Support your conclusion.

---

a.) Continuity at $x=1 \Rightarrow \lim_{x\to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1) = 1$

$\lim_{x\to 1^+} f(x) = 1 + k + p = 1 \Rightarrow k + p = 0$



Differentiability at $x=1 \Rightarrow$
$\lim_{x\to 1^-} f'(x) = \lim_{x\to 1^+} f'(x)$

$$f'(x) = \begin{cases} 2-2x, & x \le 1 \\ 2x+k, & x > 1 \end{cases}$$

$\lim_{x\to 1^-} f'(x) = 0$ and $\lim_{x\to 1^+} f'(x) = 2+k \Rightarrow 2+k=0$ and $k = -2$

$k+p=0 \Rightarrow p=2$ $\boxed{k = -2 \text{ and } p = 2}$

b.) $f'(x) = \left\{ \begin{array}{c} 2-2x > 0 \ \forall x < 1 \\ 0 \text{ when } x = 1 \\ 2x-2 > 0 \ \forall x > 1 \end{array} \right\} \Rightarrow \boxed{f \text{ is increasing } \forall x}$

c.) $f''(x) = \begin{cases} -2 & \text{when } x < 1 \\ 2 & \text{when } x > 1 \end{cases}$ OR $f''(x)$: $\underset{1}{\overset{- \quad | \quad +}{\longrightarrow}} x$

f is continuous at $x=1$. Since f'' changes signs on either side of $x=1$, $(1,1)$ is a point of inflection.

1992 – BC 5

5. The length of a solid cylindrical cord of elastic material is 32 inches. A circular cross section of the cord has radius $\frac{1}{2}$ inch.

(a) What is the volume, in cubic inches, of the cord?

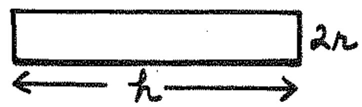
(b) The cord is stretched lengthwise at a constant rate of 18 inches per minute. Assuming that the cord maintains a cylindrical shape and a constant volume, at what rate is the radius of the cord changing one minute after the stretching begins? Indicate units of measure.

(c) A force of $2x$ pounds is required to stretch the cord x inches beyond its natural length of 32 inches. How much work is done during the first minute of stretching described in part (b)? Indicate units of measure.

---

a.) $V = \pi r^2 h = \pi \left(\frac{1}{2}\right)^2 \times 32 = 8\pi$

$\boxed{V(t) = 8\pi \text{ in}^3}$



b.) $t = 1 \Rightarrow h = 50$ and $8\pi = \pi r^2 50 \Rightarrow r = \frac{2}{5}$

$8\pi = \pi r^2 h \Rightarrow r^2 = \frac{8}{h}$ and $2r \frac{dr}{dt} = -\frac{8}{h^2} \frac{dh}{dt}$

At $t = 1$, $r = \frac{2}{5}$, $\frac{dh}{dt} = 18$, $h = 50$

$2\left(\frac{2}{5}\right) \frac{dr}{dt} = -\frac{8}{50^2}(18) \Rightarrow \frac{dr}{dt} = \frac{-9}{125}$

$\boxed{\text{When } t = 1,\ r \text{ is decreasing at the rate of } \frac{9}{125} \text{ in/min}}$

c.) $W = \int_a^b F(x)\,dx = \int_0^{18} 2x\,dx = x^2\Big]_0^{18} = 18^2$

$\boxed{\text{The work done during the first min. of stretching is 324 inch-pounds.}}$

1992 - BC 6

6. Consider the series $\sum_{n=2}^{\infty} \frac{1}{n^p \ln(n)}$, where $p \geq 0$.

(a) Show that the series converges for $p > 1$.

(b) Determine whether the series converges or diverges for $p = 1$. Show your analysis.

(c) Show that the series diverges for $0 \leq p < 1$.

---

a.) $n \geq 3$ and $p > 1 \Rightarrow n^p \ln n > n^p \Rightarrow \frac{1}{n^p \ln n} < \frac{1}{n^p}$

$\sum_3^\infty \frac{1}{n^p \ln n} < \sum_3^\infty \frac{1}{n^p}$ which converges by p-series test, $p > 1$

$\sum_3^\infty \frac{1}{n^p \ln n}$ converges by Comparison Test $\Rightarrow$

$$\boxed{\sum_2^\infty \frac{1}{n^p \ln n} \text{ converges}}$$

b.) $p = 1 \Rightarrow \sum_2^\infty \frac{1}{n^p \ln n} = \sum_2^\infty \frac{1}{n \ln n}$

Integral Test: $\int_2^\infty \frac{dx}{x \ln x} = \int_{x=2}^{\infty} \frac{du}{u}$ where $u = \ln x$, $du = \frac{dx}{x}$

$\lim_{b \to \infty} \int_{x=2}^{b} \frac{du}{u} = \lim_{b \to \infty} \ln(\ln x)\Big]_2^b = \lim_{b \to \infty} [\ln(\ln b) - \ln(\ln 2)] = \infty$

$$\boxed{\sum_2^\infty \frac{1}{n \ln n} \text{ diverges by Integral Test}}$$

c.) $0 \leq p \leq 1 \Rightarrow n^p < n \Rightarrow n^p \ln n < n \ln n$, $n \geq 2$

$\sum_2^\infty \frac{1}{n^p \ln n} > \sum_2^\infty \frac{1}{n \ln n}$ which diverges (part b)

$$\boxed{\sum_2^\infty \frac{1}{n^p \ln n} \text{ diverges by Comparison Test}}$$

1993 – AB 1

1. Let f be the function given by $f(x) = x^3 - 5x^2 + 3x + k$, where k is a constant.

   (a) On what intervals is f increasing?

   (b) On what intervals is the graph of f concave downward?

   (c) Find the value of k for which f has 11 as its relative minimum.

$f(x) = x^3 - 5x^2 + 3x + k$, k = constant

a.) f increases when $f'(x) \overset{?}{\geq} 0$

$f'(x) = 3x^2 - 10x + 3 = (3x-1)(x-3)$

		$x < \frac{1}{3}$	$\frac{1}{3} < x < 3$	$x > 3$
$f'(x)$:	$3x-1$:	−	+	+
	$x-3$:	−	−	+
$f'(x)$:		+	−	+
f-graph		Incr.	Decr.	Incr.

$\boxed{f \text{ is increasing when } x \leq \frac{1}{3} \text{ or } x \geq 3}$

b.) Graph of f is concave downward when $f''(x) < 0$

$f''(x) = 6x - 10$

	$x < \frac{5}{3}$	$x > \frac{5}{3}$
$f''(x)$:	−	+

$\boxed{f \text{ is concave downward when } x < \frac{5}{3}}$

c.) f has a relative min. when $x = 3$ (see a.)

$f(3) = 27 - 45 + 9 + k = 11 \Rightarrow k = 20$

$\boxed{f \text{ has a rel. min. of } 11 \text{ when } k = 20}$

1993 - AB 2

2. A particle moves on the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = 2te^{-t}$.

(a) Find the acceleration of the particle at $t = 0$.

(b) Find the velocity of the particle when its acceleration is 0.

(c) Find the total distance traveled by the particle from $t = 0$ to $t = 5$.

---

a) Given: $x(t) = 2te^{-t}, \quad t \geq 0$

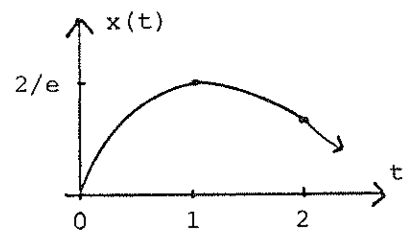
$v(t) = \frac{dx}{dt} = 2[te^{-t}(-1) + e^{-t}]$

$v(t) = 2e^{-t}(1-t)$

$a(t) = 2[e^{-t}(-1) + (1-t)e^{-t}(-1)]$

$a(t) = 2e^{-t}(t-2)$

$\boxed{a(0) = -4}$



b) $a(t) = 0$ when $t = 2$

$v(t) = 2e^{-t}(1-t)$

$\boxed{v(2) = 2e^{-2}(-1) = -2e^{-2}}$

c) The particle changes direction when $t = 1$

$x_0 = 0$ $\quad > \frac{2}{e}$

$x_1 = 2e^{-1}$ $\quad > \frac{2}{e} - \frac{10}{e^5}$

$x_5 = 10e^{-5}$

$\boxed{\text{Total distance} = \frac{4}{e} - \frac{10}{e^5} \text{ OR } \frac{4e^4 - 10}{e^5}}$

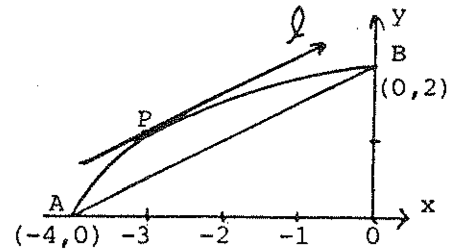
1993-AB3, BC1

3. Consider the curve $y^2 = 4 + x$ and chord AB joining points $A(-4, 0)$ and $B(0, 2)$ on the curve.

(a) Find the x- and y-coordinates of the point on the curve where the tangent line is parallel to chord AB.

(b) Find the area of the region R enclosed by the curve and chord AB.

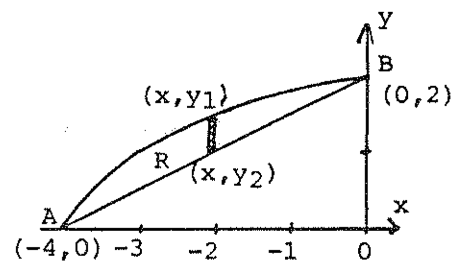
(c) Find the volume of the solid generated when the region R, defined in part (b), is revolved about the x-axis.



a.) $y^2 = 4 + x \Rightarrow 2y\frac{dy}{dx} = 1;\ \frac{dy}{dx} = \frac{1}{2y}$

Slope $\overleftrightarrow{AB} = \frac{1}{2} = \frac{1}{2y} \Rightarrow y = 1,\ x = -3$

$\boxed{\text{Tangent } \ell \parallel \overleftrightarrow{AB} \text{ at } (-3, 1)}$



b.) "Vertical strips"

$dA = (y_1 - y_2)\,dx$

$A = \int_{-4}^{0} \left(\sqrt{4+x} - \left(\frac{x}{2} + 2\right)\right) dx$

$A = \left[\frac{2}{3}(4+x)^{3/2} - \frac{x^2}{4} - 2x\right]_{-4}^{0}$ Eq. of $\overleftrightarrow{AB}$: $y = \frac{x}{2} + 2$

$A = \left[\frac{2}{3}(4)^{3/2} - \left(-\frac{16}{4} + 8\right)\right] = \frac{16}{3} + 4 - 8 = \frac{4}{3}$

$\boxed{\text{Area of region } R = \frac{4}{3}}$

c.) "Washers" (See b.) diagram)

$dV = \pi(R^2 - r^2)h = \pi(y_1^2 - y_2^2)\,dx$

$V = \pi\int_{-4}^{0}\left[(4+x) - \left(\frac{x}{2}+2\right)^2\right]dx = \pi\int_{-4}^{0}\left(4 + x - \frac{x^2}{4} - 2x - 4\right)dx$

$V = \pi\int_{-4}^{0}\left(-\frac{x^2}{4} - x\right)dx = \pi\left[-\frac{x^3}{12} - \frac{x^2}{2}\right]_{-4}^{0} = -\pi\left(\frac{64}{12} - \frac{16}{2}\right)$

$\boxed{V = \frac{8\pi}{3}}$

1993 – AB 4, BC 3

4. Let f be the function defined by $f(x) = \ln(2 + \sin x)$ for $\pi \le x \le 2\pi$.

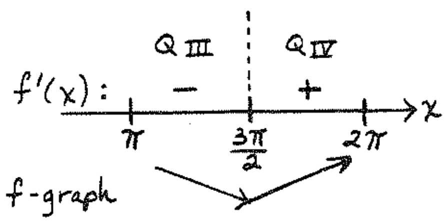
(a) Find the absolute maximum value and the absolute minimum value of f. Show the analysis that leads to your conclusion.

(b) Find the x-coordinate of each inflection point on the graph of f. Justify your answer.

---

a.) $f(x) = \ln(2+\sin x) \ge 0$ since $2 + \sin x \ge 1$; $\pi \le x \le 2\pi$

$f'(x) = \dfrac{\cos x}{2+\sin x}$; $f'(x) = 0$ $\qquad \cos x = 0 \Rightarrow x = \dfrac{3\pi}{2}$



x	$f(x)$
π	$\ln 2 = .693^+$
$\frac{3\pi}{2}$	$\ln 1 = 0$
2π	$\ln 2 = .693^+$

$\boxed{\begin{array}{l} f \text{ has an abs. min. of } \ln 1 = 0 \\ f \text{ has an abs. max. of } \ln 2 \approx .693 \end{array}}$

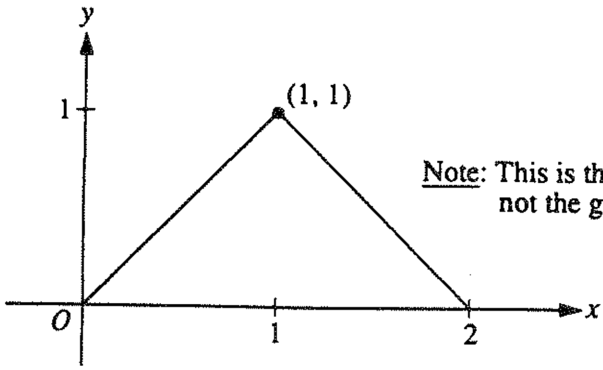
b.) Inflection points occur at points where the direction of concavity changes.

$$f''(x) = \frac{(2+\sin x)(-\sin x) - (\cos x)(\cos x)}{(2+\sin x)^2} = \frac{-2\sin x - 1}{(2+\sin x)^2}$$

$f''(x) = 0$ when $\sin x = -\frac{1}{2} \Rightarrow x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$

$\boxed{f \text{ has inflection points at } x = \frac{7\pi}{6}, \frac{11\pi}{6}}$

1993 - AB 5



Note: This is the graph of the derivative of f, not the graph of f.

5. The figure above shows the graph of f', the derivative of a function f. The domain of f is the set of all x such that $0 < x < 2$.

   (a) Write an expression for $f'(x)$ in terms of x.

   (b) Given that $f(1) = 0$, write an expression for $f(x)$ in terms of x.

   (c) In the xy-plane provided below, sketch the graph of $y = f(x)$.

---

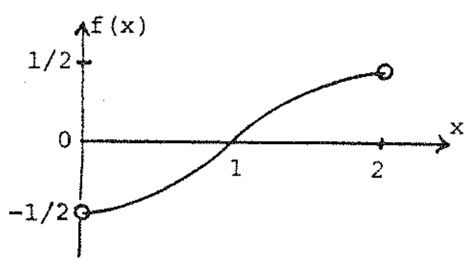
a.) $\boxed{f'(x) = \begin{cases} x, & 0 < x \le 1 \\ 2-x, & 1 \le x < 2 \end{cases} \quad \underline{\text{OR}} \quad 1 - |x-1|, \; 0 < x < 2}$

b.) $0 < x \le 1: \; f(x) = \frac{x^2}{2} + C_1$ and $f(1) = 0 \Rightarrow C_1 = -\frac{1}{2}$

$1 \le x < 2: \; f(x) = 2x - \frac{x^2}{2} + C_2; \; f(1) = 0 = 2 - \frac{1}{2} + C_2 \Rightarrow C_2 = -\frac{3}{2}$

$$\boxed{f(x) = \begin{cases} \frac{x^2}{2} - \frac{1}{2}, & 0 < x \le 1 \\ 2x - \frac{x^2}{2} - \frac{3}{2}, & 1 \le x < 2 \end{cases}}$$

c.)



1993 - AB 6

6. Let $P(t)$ represent the number of wolves in a population at time t years, when $t \geqq 0$. The population $P(t)$ is increasing at a rate directly proportional to $800 - P(t)$, where the constant of proportionality is k.

(a) If $P(0) = 500$, find $P(t)$ in terms of t and k.

(b) If $P(2) = 700$, find k.

(c) Find $\lim_{t \to \infty} P(t)$.

---

a.) $\frac{d(P(t))}{dt} = k(800 - P(t))$

$\frac{d(P(t))}{800 - P(t)} = k\,dt$ | $u = 800 - P(t)$, $du = -d(P(t))$

$-\frac{du}{u} = k\,dt \Rightarrow \ln u = -(kt + c) \Rightarrow$

$u = e^{-kt + C_1} = Ce^{-kt}$

$800 - P(t) = Ce^{-kt}$

$t = 0 \Rightarrow P(t) = 500 \Rightarrow 800 - 500 = C$ or $C = 300$

$$\boxed{P(t) = 800 - 300e^{-kt}}$$

b.) $P(2) = 700 \Rightarrow 700 = 800 - 300e^{-2k}$

$300e^{-2k} = 100 \Rightarrow e^{-2k} = \frac{1}{3} \Rightarrow -2k = -\ln 3$

$$\boxed{k = \frac{\ln 3}{2}}$$

c.) $\lim_{t \to \infty} P(t) = \lim_{t \to \infty} \left[800 - 300e^{-\frac{\ln 3}{2}t}\right]$

$= 800 - \lim_{t \to \infty} \frac{300}{e^{\frac{\ln 3}{2}t}} = 800$

$$\boxed{\lim_{t \to \infty} P(t) = 800}$$

1993 – BC2

2. The position of a particle at any time $t \geq 0$ is given by $x(t) = t^2 - 3$ and $y(t) = \frac{2}{3}t^3$.

(a) Find the magnitude of the velocity vector at $t = 5$.

(b) Find the total distance traveled by the particle from $t = 0$ to $t = 5$.

(c) Find $\frac{dy}{dx}$ as a function of x.

---

a.) $x = t^2 - 3 \qquad \frac{dx}{dt} = 2t \qquad t \geq 0$

$y = \frac{2}{3}t^3 \qquad \frac{dy}{dt} = 2t^2$

$$|\vec{v}| = \sqrt{(dx/dt)^2 + (dy/dt)^2}$$

$$= \sqrt{4t^2 + 4t^4} = 2t\sqrt{1+t^2}$$

$$\boxed{|\vec{v}|_{t=5} = 10\sqrt{26}}$$

b.) $$s = \int_0^5 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^5 2t\sqrt{1+t^2}\, dt = \frac{2}{3}(1+t^2)^{3/2}\Big]_0^5$$

$$\boxed{s = \frac{2}{3}\left(26^{3/2} - 1\right)}$$

c.) $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t^2}{2t} = t = \sqrt{x+3}$$

$$\boxed{\frac{dy}{dx} = \sqrt{x+3}}$$

1993 – BC 4

4. Consider the polar curve $r = 2\sin(3\theta)$ for $0 \le \theta \le \pi$.

(a) In the xy-plane provided below, sketch the curve.

Note: The xy-plane is provided in the pink test booklet only.

(b) Find the area of the region inside the curve.

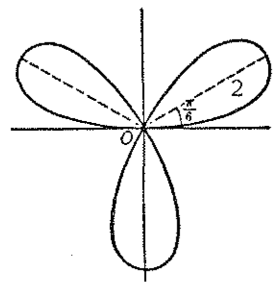
(c) Find the slope of the curve at the point where $\theta = \frac{\pi}{4}$.

---

a.) $r = 2\sin 3\theta$, $0 \le \theta \le \pi$ (Three – Leaved Rose)

One-leaf:

θ	0	$\pi/18$	$\pi/6$	$5\pi/18$	$\pi/3$
3θ	0	$\pi/6$	$\pi/2$	$5\pi/6$	π
r	0	1	2	1	0



b.) $\text{Area} = \int_{\theta=0}^{\pi} \frac{r^2\,d\theta}{2} = \int_0^{\pi} \frac{4\sin^2 3\theta\,d\theta}{2} = 2\int_0^{\pi} \sin^2 3\theta\,d\theta$

$= 2\int_0^{\pi} \frac{(1-\cos 6\theta)}{2}\,d\theta = \left[\theta - \frac{\sin 6\theta}{6}\right]_0^{\pi} = \pi$

$\boxed{\text{Area} = \pi}$

c.) $\text{Slope} = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$ where $y = r\sin\theta$, $x = r\cos\theta$

$$\frac{dy/d\theta}{dx/d\theta} = \frac{d(r\sin\theta)/d\theta}{d(r\cos\theta)/d\theta} = \frac{r\cos\theta + dr/d\theta\,\sin\theta}{-r\sin\theta + dr/d\theta\,\cos\theta}$$

$r = 2\sin 3\theta \Rightarrow dr/d\theta = 6\cos 3\theta$

$$\frac{dy}{dx} = \frac{2\sin 3\theta\cos\theta + 6\cos 3\theta\sin\theta}{-2\sin 3\theta\sin\theta + 6\cos 3\theta\cos\theta}$$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/4} = \frac{2\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} - 6\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}}{-2\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} - 6\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}} = \frac{1-3}{-1-3} = \boxed{\frac{1}{2}}$$

1993 – BC 5

5. Let f be the function given by $f(x) = e^{\frac{x}{2}}$.

(a) Write the first four nonzero terms and the general term for the Taylor series expansion of $f(x)$ about $x = 0$.

(b) Use the result from part (a) to write the first three nonzero terms and the general term of the series expansion about $x = 0$ for $g(x) = \dfrac{e^{\frac{x}{2}} - 1}{x}$.

(c) For the function g in part (b), find $g'(2)$ and use it to show that $\displaystyle\sum_{n=1}^{\infty} \frac{n}{4(n+1)!} = \frac{1}{4}$.

---

a.) $$e^{u} = \sum_{n=0}^{\infty} \frac{u^n}{n!} = 1 + u + \frac{u^2}{2} + \frac{u^3}{3!} + \ldots + \frac{u^n}{n!} + \ldots$$

$$\boxed{e^{x/2} = \sum_{n=0}^{\infty} \frac{x^n}{2^n n!} = 1 + \frac{x}{2} + \frac{x^2}{2^2 \cdot 2!} + \frac{x^3}{2^3 \cdot 3!} + \ldots + \frac{x^n}{2^n \cdot n!} + \ldots}$$

b.) $$\boxed{g(x) = \frac{e^{x/2} - 1}{x} = \frac{1}{2} + \frac{x}{2^2 \cdot 2!} + \frac{x^2}{2^3 \cdot 3!} + \ldots + \frac{x^{n-1}}{2^n \cdot n!} + \ldots = \sum_{n=1}^{\infty} \frac{x^{n-1}}{2^n \cdot n!}}$$

c.) $$g'(x) = \frac{1}{2^2 2!} + \frac{2x}{2^3 \cdot 3!} + \frac{3x^2}{2^4 \cdot 4!} + \ldots + \frac{nx^{n-1}}{2^{n+1} \cdot (n+1)!} + \ldots = \sum_{1}^{\infty} \frac{nx^{n-1}}{2^{n+1}(n+1)!}$$

$$g'(2) = \frac{1}{2^2 \cdot 2!} + \frac{2}{2^2 \cdot 3!} + \frac{3}{2^2 \cdot 4!} + \ldots + \frac{n}{2^2 \cdot (n+1)!} + \ldots = \sum_{1}^{\infty} \frac{n}{2^2 (n+1)!}$$

$$g(x) = \frac{e^{x/2} - 1}{x} \Rightarrow g'(x) = \frac{x\left(\frac{1}{2} e^{x/2}\right) - \left(e^{x/2} - 1\right)}{x^2} \Rightarrow$$

$$g'(2) = \frac{e - (e-1)}{4} = \frac{1}{4}$$

$$\boxed{\therefore \sum_{1}^{\infty} \frac{n}{4(n+1)!} = \frac{1}{4}}$$

1993 – BC 6

6. Let f be a function that is differentiable throughout its domain and that has the following properties.

(i) $f(x+y) = \dfrac{f(x)+f(y)}{1-f(x)f(y)}$ for all real numbers x, y, and $x+y$ in the domain of f.

(ii) $\lim_{h\to 0} f(h) = 0$ (iii) $\lim_{h\to 0} \dfrac{f(h)}{h} = 1$

(a) Show that $f(0) = 0$.

(b) Use the definition of the derivative to show that $f'(x) = 1 + [f(x)]^2$. Indicate clearly where properties (i), (ii), and (iii) are used.

(c) Find $f(x)$ by solving the differential equation in part (b).

---

a.) $x = y = 0 \Rightarrow f(0) = \dfrac{f(0)+f(0)}{1-[f(0)]^2}$, $[f(0)]^2 \neq 1$ (by (i))

$2f(0) = f(0)\left(1-[f(0)]^2\right) = f(0) - [f(0)]^3 \Rightarrow f(0) + [f(0)]^3 = 0$

$f(0)\left(1+[f(0)]^2\right) = 0$. Since $1+[f(0)]^2 \geq 1$, $f(0) = 0$

b.) $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\left[\dfrac{\frac{f(x)}{h}+\frac{f(h)}{h}}{1-f(x)f(h)} - \dfrac{f(x)}{h}\right]$ (by (i))

$= \lim_{h\to 0} \dfrac{\frac{f(x)}{h}+\frac{f(h)}{h}-\frac{f(x)}{h}\left(1-f(x)f(h)\right)}{1-f(x)f(h)} = \lim_{h\to 0} \dfrac{\frac{f(h)}{h}+[f(x)]^2\frac{f(h)}{h}}{1-f(x)f(h)}$

$\boxed{f'(x) = 1+[f(x)]^2}$ (from (ii) and (iii))

c.) $f'(x) = \dfrac{d(f(x))}{dx} = 1+[f(x)]^2 \Rightarrow \dfrac{d(f(x))}{1+[f(x)]^2} = dx \Rightarrow$

$\arctan f(x) = x + C$

or $f(x) = \tan(x+C)$

Since $f(0) = 0$, $f(0) = \tan C \Rightarrow C = 0$ or $n\pi$, $n \in I$

$\boxed{f(x) = \tan x \text{ or } \tan(x+n\pi),\ n \in I}$

1994 - AB1

1. Let f be the function given by $f(x) = 3x^4 + x^3 - 21x^2$.

   (a) Write an equation of the line tangent to the graph of f at the point $(2, -28)$.

   (b) Find the absolute minimum value of f. Show the analysis that leads to your conclusion.

   (c) Find the x-coordinate of each point of inflection on the graph of f. Show the analysis that leads to your conclusion.

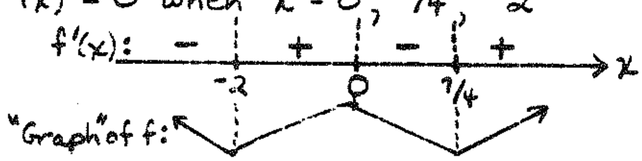
a.) Equation of tangent line: $y+28 = m(x-2)$ where $m = f'(2)$

$f'(x) = 12x^3 + 3x^2 - 42x$; $f'(2) = 96 + 12 - 84 = 24$

$\boxed{y + 28 = 24(x-2) \text{ or } y = 24x - 76}$

b.) $f'(x) = 3x(4x^2 + x - 14) = 3x(4x-7)(x+2)$

$f'(x) = 0$ when $x = 0, \ 7/4, \ -2$



Candidates for absol. min. are at $x = -2$ and at $x = 7/4$.

$f(-2) = -44$ and $f(7/4) \approx -30.816$

$\boxed{f \text{ has an absolute minimum of } -44}$

c.) $f''(x) = 36x^2 + 6x - 42 = 6(6x+7)(x-1)$

$f''(x) = 0$ when $x = -7/6$ or 1

$f''(x)$: + | − | + → x

c.u. $-7/6$ c.D. 1 c.u.

f changes direction of concavity at $x = -7/6$ and at $x = 1$

$\boxed{f \text{ has points of inflection at } x = -7/6 \text{ and at } x = 1}$

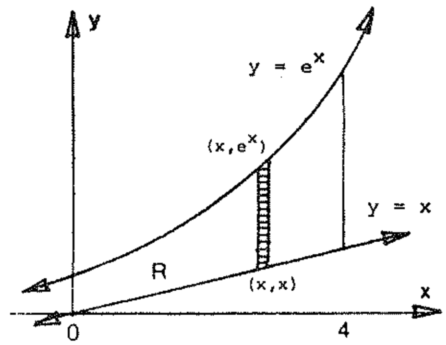
1994 - AB2, BC1

2. Let R be the region enclosed by the graphs of $y = e^x$, $y = x$, and the lines $x = 0$ and $x = 4$.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is revolved about the x-axis.

(c) Set up, but do not integrate, an integral expression in terms of a single variable for the volume of the solid generated when R is revolved about the y-axis.



a.) $dA = (e^x - x)\,dx$

$A = \int_0^4 (e^x - x)\,dx$

$A = e^x - \frac{x^2}{2}\Big]_0^4 = e^4 - 8 - (1-0)$

$\boxed{A = e^4 - 9}$

b.) "Washers"

$dV_x = \pi (R^2 - r^2)\,dx$

$V_x = \pi \int_0^4 (R^2 - r^2)\,dx = \pi \int_0^4 (e^{2x} - x^2)\,dx$

$V_x = \pi\left[\frac{e^{2x}}{2} - \frac{x^3}{3}\right]_0^4 = \pi\left[\frac{e^8}{2} - \frac{64}{3} - (\frac{1}{2} - 0)\right]$

$\boxed{V_x = \pi\left(\frac{e^8}{2} - \frac{131}{6}\right)}$

c.) "Cylindrical Shells"

$dV_y = 2\pi r h\,dx$

$\boxed{V_y = 2\pi \int_0^4 x(e^x - x)\,dx}$

1994 - AB3

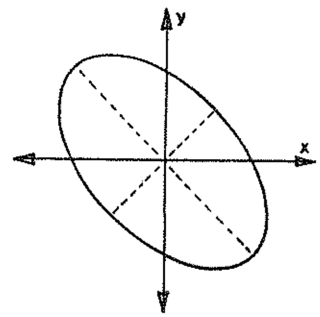
3. Consider the curve defined by $x^2 + xy + y^2 = 27$.

(a) Write an expression for the slope of the curve at any point (x, y).

(b) Determine whether the lines tangent to the curve at the x-intercepts of the curve are parallel. Show the analysis that leads to your conclusion.

(c) Find the points on the curve where the lines tangent to the curve are vertical.

a) Slope: $2x + x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0$



$\frac{dy}{dx}(x+2y) = -(2x+y)$

$$\boxed{\frac{dy}{dx} = -\frac{(2x+y)}{x+2y}}$$

b) When $y=0$, $x = \pm 3\sqrt{3}$
and $\frac{dy}{dx} = \frac{-2x}{x} = -2$

$\boxed{\text{The tangent lines at the x intercepts are //}}$

c) $\frac{dy}{dx} = \frac{-(2x+y)}{x+2y}$ is not defined when $x+2y=0$ or $y = \frac{-x}{2}$

Then $x^2 + x\left(-\frac{x}{2}\right) + \left(-\frac{x}{2}\right)^2 = 27$

or $x^2 - \frac{x^2}{2} + \frac{x^2}{4} = 27 \Rightarrow \frac{3}{4}x^2 = 27 \Rightarrow x^2 = 36$

$x = \pm 6$ and $y = \mp 3$

$\boxed{\text{The curve has vertical tangents at } (6,-3) \text{ and } (-6,3)}$

1994 - AB4

4. A particle moves along the x-axis so that at any time $t > 0$ its velocity is given by $v(t) = t \ln t - t$. At time $t = 1$, the position of the particle is $x(1) = 6$.

(a) Write an expression for the acceleration of the particle.

(b) For what values of t is the particle moving to the right?

(c) What is the minimum velocity of the particle? Show the analysis that leads to your conclusion.

(d) Write an expression for the position $x(t)$ of the particle.

a.) $a(t) = \frac{dv}{dt} = \frac{t}{t} + \ln t - 1 = \ln t, \; t > 0$

$$\boxed{a(t) = \ln t, \; t > 0}$$

b.) Particle moves right when $v(t) > 0$

$v(t) = t(\ln t - 1), \; t > 0$

$\ln t - 1$: $-$ on $(0, e)$, $+$ for $t > e$

$$\boxed{\text{Particle moves right when } t > e}$$

c.) v min: $a(t) = v'(t) = \ln t \; ; \; v'(t) = 0 \Rightarrow t = 1$

$v'(t)$: $-$ on $(0, 1)$, $+$ for $t > 1$

"Graph" v

$$\boxed{v \text{ has absol. min. of } -1}$$

d.) $x(t) = \int v \, dt = \int t \ln t \, dt - \int t \, dt$

$= \frac{t^2}{2} \ln t - \int \frac{t}{2} \, dt - \frac{t^2}{2} + C$

$= \frac{t^2}{2} \ln t - \frac{3}{4} t^2 + C$

$u = \ln t$	$dv = t \, dt$
$du = \frac{1}{t} \, dt$	$v = \frac{t^2}{2}$

$x(1) = 6 = 0 - \frac{3}{4} + C \Rightarrow C = \frac{27}{4}$

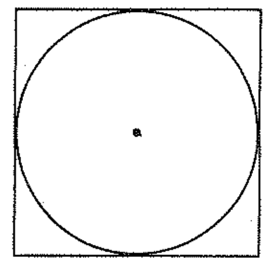
$$\boxed{x(t) = \frac{t^2}{2} \ln t - \frac{3}{4} t^2 + \frac{27}{4}}$$

1994 - AB5, BC2

5. A circle is inscribed in a square as shown in the figure above. The circumference of the circle is increasing at a constant rate of 6 inches per second. As the circle expands, the square expands to maintain the condition of tangency. (Note: A circle with radius r has circumference $C = 2\pi r$ and area $A = \pi r^2$.)

   (a) Find the rate at which the perimeter of the square is increasing. Indicate units of measure.

   (b) At the instant when the area of the circle is 25π square inches, find the rate of increase in the area enclosed between the circle and the square. Indicate units of measure.



Given: $\frac{d(2\pi r)}{dt} = 2\pi \frac{dr}{dt} = 6 \Rightarrow \frac{dr}{dt} = \frac{3}{\pi}$

a) $P = 8r$; $\frac{dP}{dt} = 8\frac{dr}{dt}$

$$\boxed{\frac{dP}{dt} = 8 \times \frac{3}{\pi} = \frac{24}{\pi} \text{ (in./sec.)}}$$

b) $A = 4r^2 - \pi r^2 = (4-\pi)r^2$

Find $\frac{dA}{dt}$ when $\pi r^2 = 25\pi \Rightarrow r = 5$

$\frac{dA}{dt} = (4-\pi)2r\frac{dr}{dt}$; $\left.\frac{dA}{dt}\right|_{r=5} = (4-\pi)(10)\left(\frac{3}{\pi}\right)$

$$\boxed{\left.\frac{dA}{dt}\right|_{r=5} = \left(\frac{120}{\pi} - 30\right) \text{ in.}^2\text{/sec.}}$$

1994 – AB6

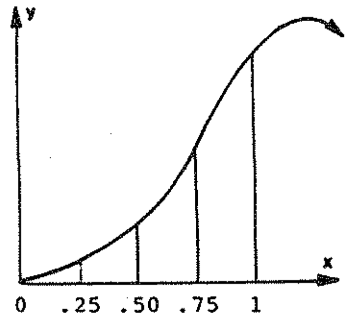
6. Let $F(x) = \int_0^x \sin(t^2)dt$ for $0 \leq x \leq 3$.

(a) Use the trapezoidal rule with four equal subdivisions of the closed interval [0, 1] to approximate $F(1)$.

(b) On what intervals is F increasing?

(c) If the average rate of change of F on the closed interval [1, 3] is k, find $\int_1^3 \sin(t^2)dt$ in terms of k.

a.) $F(1) = \int_0^1 \sin(t^2)dt$



$$\boxed{F(1) \approx \frac{1}{8}\left[2\left(\sin(.25)^2 + \sin(.5)^2 + \sin(.75)^2\right) + \sin 1\right] \approx .316}$$

b.) F increases when $\sin(t^2)$ is positive

$\sin t^2 > 0$ when $0 < t^2 < \pi$ or $2\pi < t^2 < 9$

or when $0 < t < \sqrt{\pi}$ or $\sqrt{2\pi} < t < 3$

$$\boxed{F \text{ increases on } [0, \sqrt{\pi}] \text{ or } [\sqrt{2\pi}, 3]}$$

c.) $\dfrac{F(3) - F(1)}{3 - 1} = \dfrac{\int_0^3 \sin(t^2)dt - \int_0^1 \sin(t^2)dt}{2} = \dfrac{\int_1^3 \sin(t^2)dt}{2} = k$

$$\boxed{\int_1^3 \sin(t^2)\,dt = 2k}$$

1994 - BC 3

3. A particle moves along the graph of $y = \cos x$ so that the x-component of acceleration is always 2. At time $t = 0$, the particle is at the point $(\pi, -1)$ and the velocity vector of the particle is $(0,0)$.

(a) Find the x- and y- coordinates of the position of the particle in terms of t.

(b) Find the speed of the particle when its position is $(4, \cos 4)$.

a.) $\frac{d(dx/dt)}{dt} = 2 \Rightarrow \int d(dx/dt) = \int 2\,dt \Rightarrow \frac{dx}{dt} = 2t + C_1$

$t = 0 \Rightarrow \frac{dx}{dt} = 0$ and $C_1 = 0$; $\frac{dx}{dt} = 2t \Rightarrow x(t) = t^2 + C_2$

$x(0) = \pi \Rightarrow C_2 = \pi$ and $x(t) = t^2 + \pi$

$y(t) = \cos(t^2 + \pi) = -\cos(t^2)$

$\boxed{x(t) = t^2 + \pi; \quad y(t) = -\cos(t^2)}$

b.) Find speed when position of particle is $(4, \cos 4)$

$x(t) = t^2 + \pi = 4 \Rightarrow t = \sqrt{4-\pi}$

Speed $= |\vec{v}| = \sqrt{(dx/dt)^2 + (dy/dt)^2}$ when $t = \sqrt{4-\pi}$

$\frac{dx}{dt} = 2t \Rightarrow \left.\frac{dx}{dt}\right|_{t=\sqrt{4-\pi}} = 2\sqrt{4-\pi}$

$\frac{dy}{dt} = 2t\sin(t^2) \Rightarrow \left.\frac{dy}{dt}\right|_{t=\sqrt{4-\pi}} = 2\sqrt{4-\pi}\,\sin(4-\pi) = -2\sqrt{4-\pi}\,\sin 4$

$|\vec{v}| = \sqrt{(2\sqrt{4-\pi})^2 + (-2\sqrt{4-\pi}\,\sin 4)^2}$

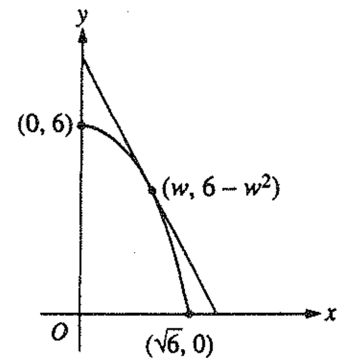
$\boxed{|\vec{v}| = 2\sqrt{4-\pi}\,\sqrt{1+\sin^2 4}}$

1994 – BC 4

4. Let $f(x) = 6 - x^2$. For $0 < w < \sqrt{6}$, let $A(w)$ be the area of the triangle formed by the coordinate axes and the line tangent to the graph of f at the point $(w, 6 - w^2)$. (See figure above.)

(a) Find $A(1)$.

(b) For what value of w is $A(w)$ a minimum?



Note: Figure not drawn to scale.

Given: $f(x) = 6 - x^2, \quad 0 < w < \sqrt{6}$

$A(w) = \text{Area } \Delta OMP$

a) Find $A(1)$ when $(w, 6-w^2) = (1, 5)$

eq. $\overleftrightarrow{MP}$: $y - (6-w^2) = m(x-w)$

where $m = f'(w) = -2w$

$y - (6-w^2) = -2w(x-w)$

y intercept $= 6+w^2$; x intercept $= \dfrac{6+w^2}{2w}$

$$\boxed{A(1) = \frac{1}{2} \times 7 \times \frac{7}{2} = \frac{49}{4}}$$

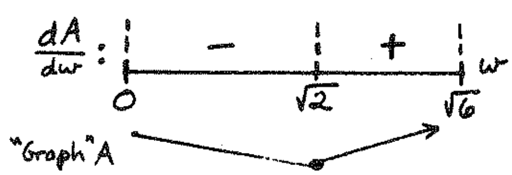
b) $A(w) = \frac{1}{2}(6+w^2)\dfrac{(6+w^2)}{2w} = \dfrac{1}{4}\dfrac{(6+w^2)^2}{w}$

$$\frac{d(A(w))}{dw} = \frac{1}{4}\frac{[(w \cdot 2(6+w^2)(2w) - (6+w^2)^2]}{w^2}$$

$$= \frac{1}{4w^2}[(6+w^2)(4w^2 - (6+w^2))]$$

$$= \frac{1}{4w^2}(6+w^2)(3w^2-6)$$

$\dfrac{dA}{dw} = 0$ when $w = \sqrt{2}$



$$\boxed{A(w) \text{ is an absolute minimum when } w = \sqrt{2}}$$

1994 – BC5

5. Let f be the function given by $f(x) = e^{-2x^2}$.

(a) Find the first four nonzero terms and the general term of the power series for $f(x)$ about $x = 0$.

(b) Find the interval of convergence of the power series for $f(x)$ about $x = 0$. Show the analysis that leads to your conclusion.

(c) Let g be the function given by the sum of the first four nonzero terms of the power series for $f(x)$ about $x = 0$. Show that $|f(x) - g(x)| < 0.02$ for $-0.6 \le x \le 0.6$.

a.) $e^u = 1 + \frac{u}{1!} + \frac{u^2}{2!} + \frac{u^3}{3!} + \ldots + \frac{u^n}{n!} + \ldots$, converges for $\forall u$

$$\boxed{e^{-2x^2} = 1 - \frac{2x^2}{1!} + \frac{(2x^2)^2}{2!} - \frac{(2x^2)^3}{3!} + \ldots + \frac{(-2x^2)^n}{n!} + \ldots}$$

b.)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}x^{2n+2}}{(n+1)!} \cdot \frac{n!}{2^n x^{2n}}\right| = \lim_{n\to\infty} \frac{2x^2}{n+1} = 0,\ \forall x$$

Series converges for all real x

$$\boxed{\text{Interval of convergence is all real numbers}}$$

c.) $f(x)$ is a strictly alternating convergent series. The maximum error which occurs when n terms of the series are summed to approximate $f(x)$ is less than the absolute value of the $(n+1)$st term of the series.

$$|f(x) - g(x)| < \frac{(2x^2)^4}{4!}, \quad -.6 < x < .6$$

Maximum value of $\frac{(2x^2)^4}{4!}$ occurs when $x = .6$

$$|f(x) - g(x)| < \frac{16(.6)^8}{4!} < .0112 < .02$$

Q.E.D.

1994 - BC6

6. Let f and g be functions that are differentiable for all real numbers x and that have the following properties.

(i) $f'(x) = f(x) - g(x)$

(ii) $g'(x) = g(x) - f(x)$

(iii) $f(0) = 5$

(iv) $g(0) = 1$

(a) Prove that $f(x) + g(x) = 6$ for all x.

(b) Find $f(x)$ and $g(x)$. Show your work.

---

a.) Add (i) and (ii): $f'(x) + g'(x) = 0 \Rightarrow \int f'(x)\,dx + \int g'(x)\,dx = 0$

Then $f(x) + g(x) = C, \ \forall x$. Take $x = 0$.

$f(0) + g(0) = C$ or $5 + 1 = C$

$$\boxed{f(x) + g(x) = 6}$$

b.) From (i): $f'(x) = f(x) - g(x)$

Since $g(x) = 6 - f(x)$, $f'(x) = f(x) - (6 - f(x))$

$f'(x) = 2f(x) - 6$ or $\frac{dy}{dx} = 2(y-3)$

$\frac{dy}{y-3} = 2dx \Rightarrow \ln|y-3| = 2x + C_1$

$y - 3 = e^{2x+C_1}$ or $e^{2x}e^{C_1}$. Let $C_2 = e^{C_1}$

Then $y = C_2 e^{2x} + 3$ or $f(x) = C_2 e^{2x} + 3$

$f(0) = 5 = C_2 + 3 \Rightarrow C_2 = 2$

$f(x) = 2e^{2x} + 3$

$g(x) = 6 - f(x) = 6 - (2e^{2x} + 3) = -2e^{2x} + 3$

$$\boxed{f(x) = 2e^{2x} + 3; \quad g(x) = -2e^{2x} + 3}$$

1995 – AB1

1. Let f be the function given by $f(x)=\dfrac{2x}{\sqrt{x^2+x+1}}$

   (a) Find the domain of f. Justify your answer.

   (b) In the viewing window provided below, sketch the graph of f.

   (c) Write an equation for each horizontal asymptote of the graph of f.

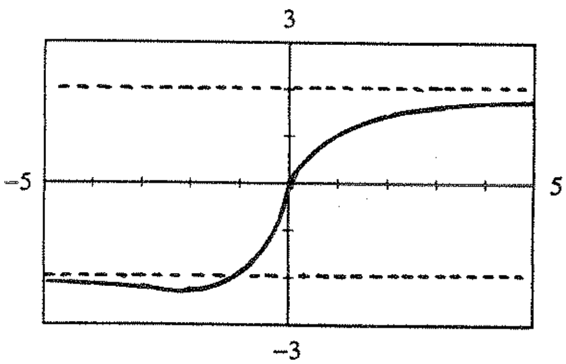
   (d) Find the range of f. Use $f'(x)$ to justify your answer.

   Note: $f'(x)=\dfrac{x+2}{(x^2+x+1)^{\frac{3}{2}}}$

(a) $x^2+x+1=\left(x+\dfrac{1}{2}\right)^2+\dfrac{3}{4}\ge\dfrac{3}{4}\quad\forall x$ or $x^2+x+1>0\quad\forall x$

$\boxed{D_f:\ x\in R\ \ or\ \ -\infty<x<\infty}$

(b)



(c) $\dfrac{2x}{\sqrt{x^2+x+1}}=\dfrac{2x}{|x|\sqrt{1+\dfrac{1}{x}+\dfrac{1}{x^2}}}$

$\lim\limits_{x\to\infty}\dfrac{2x}{x}=2\quad,\quad\lim\limits_{x\to-\infty}\dfrac{2x}{-x}=-2$

$\boxed{\text{Asymptotes: } y=2,\ y=-2}$

(d)

$f'(x)$: − (left of -2), + (right of -2)

$f(x)$: decr, incr

$\Rightarrow$ f has an absolute minimum at $x=-2$;

absolute minimum $=f(-2)=\dfrac{-4}{\sqrt{3}}$

$\boxed{R_f:\dfrac{-4}{\sqrt{3}}\le y<2\ \ or\ \ -2.309\le y<2}$

2. A particle moves along the y-axis so that its velocity at any time $t \geq 0$ is given by $v(t) = t \cos t$. At time $t = 0$, the position of the particle is $y = 3$.
   (a) For what values of t, $0 \leq t \leq 5$, is the particle moving upward?
   (b) Write an expression for the acceleration of the particle in terms of t.
   (c) Write an expression for the position $y(t)$ of the particle.
   (d) For $t > 0$, find the position of the particle the first time the velocity of the particle is zero.

(a) Particle moves upward when $v(t) = t\cos t > 0,\ t \geq 0$

$v(t)$: $\quad 0 \quad + \quad \frac{\pi}{2} \quad - \quad \frac{3\pi}{2} \quad + \quad 5$

$$\boxed{\text{Particle moves upward when } 0 < t < \tfrac{\pi}{2} \text{ or } \tfrac{3\pi}{2} < t \leq 5}$$

(b) $\boxed{a = \dfrac{dv}{dt} = -t\sin t + \cos t}$

(c) $y = \int v\,dt = \int t\cos t\,dt \qquad u = t \quad dv = \cos t$
$\qquad\qquad\qquad\qquad\qquad du = dt \quad v = \sin t$

$y(t) = t\sin t - \int \sin t\,dt = t\sin t + \cos t + C$

$y(0) = 3 = 1 + C \Rightarrow C = 2$

$\boxed{y(t) = t\sin t + \cos t + 2}$

(d) $v(t) = 0$ *and* $t > 0 \Rightarrow t = \frac{\pi}{2}$

$y(\frac{\pi}{2}) = \frac{\pi}{2}\sin\frac{\pi}{2} + \cos\frac{\pi}{2} + 2$

$\boxed{y(\tfrac{\pi}{2}) = \tfrac{\pi}{2} + 2 \approx 3.571}$

3. Consider the curve defined by $-8x^2 + 5xy + y^3 = -149$

(a) Find $\frac{dy}{dx}$.

(b) Write an equation for the line tangent to the curve at the point (4, –1).

(c) There is a number k so that the point (4.2, k) is on the curve. Using the tangent line found in part (b), approximate the value of k.

(d) Write an equation that can be solved to find the actual value of k so that the point (4.2 , k) is on the curve.

(e) Solve the equation found in part (d) for the value of k.

(a) $-16x + 5(x\frac{dy}{dx} + y) + 3y^2\frac{dy}{dx} = 0$

$$\frac{dy}{dx}(5x + 3y^2) = 16x - 5y$$

$$\boxed{\frac{dy}{dx} = \frac{16x - 5y}{5x + 3y^2}}$$

(b) $\left.\frac{dy}{dx}\right|_{(4,-1)} = \frac{64+5}{20+3} = 3$

$$\boxed{y + 1 = 3(x - 4) \quad or \quad y = 3x - 13}$$

(c) $y + 1 = 3(.2)$

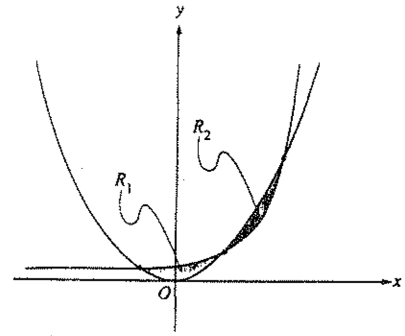
$$\boxed{y = -0.4 \quad or \quad k \approx -0.4}$$

(d) $\boxed{-8(4.2)^2 + 5k(4.2) + k^3 = -149 \quad or \quad 21k + k^3 + 7.88 = 0}$

(e) $\boxed{k \approx -.373}$

1995 – AB4 , BC2

4. The shaded regions R_1 and R_2 shown above are enclosed by the graphs of $f(x) = x^2$ and $g(x) = 2^x$.
   (a) Find the x- and y- coordinates of the three points of intersection of the graphs of f and g.
   (b) Without using absolute value, set up an expression involving one or more integrals that gives the total area enclosed by the graphs of f and g. Do not evaluate.
   (c) Without using absolute value, set up an expression involving one or more integrals that gives the volume of the solid generated by revolving the region R_1 about the line y = 5. Do not evaluate.



Note: Figure not drawn to scale.

(a) $\boxed{(-.767\,, .588\,)\;,\;(\,2, 4\,)\;,\;(\,4, 16\,)}$

(b) use (1) "vertical strips" or (2) "horizontal strips"

(1) $A = \int_{-.767}^{2} (2^x - x^2)\,dx \; + \; \int_{2}^{4} (x^2 - 2^x)\,dx,$

or

(2) $A = \int_{0}^{.588} \left(\sqrt{y} - (-\sqrt{y})\right) dy \; + \; \int_{.588}^{4} \left(\sqrt{y} - \frac{\ln y}{\ln 2}\right) dy \; + \; \int_{4}^{16} \left(\frac{\ln y}{\ln 2} - \sqrt{y}\right) dy$

(c) use (1) "washers" or (2) "cylindrical shells"

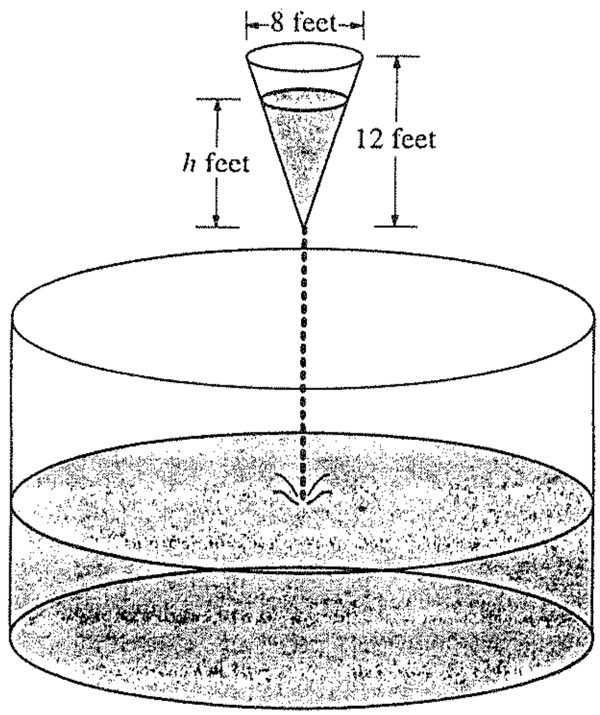
(1) $V_{y=5} = \pi \int_{-.767}^{2} \left[(5 - x^2)^2 - (5 - 2^x)^2\right] dx,$

or

(2) $V_{y=5} = 2\pi \int_{0}^{.588} (5 - y)\left(\sqrt{y} - (-\sqrt{y})\right) dy \; + \; 2\pi \int_{.588}^{4} (5 - y)\left(\sqrt{y} - \left(\frac{\ln y}{\ln 2}\right)\right) dy$

5. As shown in the figure above, water is draining from a conical tank with height 12 feet and diameter 8 feet into a cylindrical tank that has a base with area 400 π square feet. The depth h, in feet, of the water in the conical tank is changing at the rate of (h – 12) feet per minute. (The volume V of a cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$.)
   (a) Write an expression for the volume of water in the conical tank as a function of h.
   (b) At what rate is the volume of water in the conical tank changing when h = 3 ? Indicate units of measure.
   (c) Let y be the depth, in feet, of the water in the cylindrical tank. At what rate is y changing when h = 3 ? Indicate units of measure.

Given: $\frac{dh}{dt} = (h-12)\,\frac{ft}{\min}$



(a) $\frac{h}{2r} = \frac{12}{8} \Rightarrow r = \frac{h}{3}$

$$\boxed{V = \frac{\pi}{3}\cdot\frac{h^2}{9}\cdot h = \frac{\pi h^3}{27}\,(ft^3)}$$

(b) $\frac{dV}{dt} = \frac{3\pi h^2}{27}\cdot\frac{dh}{dt} = \frac{\pi h^2(h-12)}{9}\left(\frac{ft^3}{\min}\right)$

$$\boxed{\left.\frac{dV}{dt}\right|_{h=3} = \pi\frac{9(-9)}{9} = -9\pi\left(\frac{ft^3}{\min}\right)}$$

(c) $V_{cyl} = 400\pi y$

$\frac{dV_c}{dt} = 400\pi\,\frac{dy}{dt}$; $\frac{dV_c}{dt} = 9\pi\left(\frac{ft^3}{\min}\right)$ from (b)

$9\pi = 400\pi\,\frac{dy}{dt}$

$$\boxed{\frac{dy}{dt} = \frac{9}{400}\left(\frac{ft}{\min}\right)}$$

1995 – AB6

6. The graph of a differentiable function f on the closed interval [1, 7] is shown.

   Let $h(x) = \int_1^x f(t)\,dt$ for $1 \le x \le 7$.

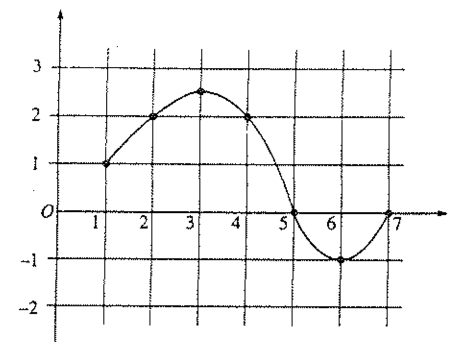
   (a) Find $h(1)$.

   (b) Find $h'(4)$.

   (c) On what interval or intervals is the graph of h concave upward? Justify your answer.

   (d) Find the value of x at which h has its minimum on the closed interval [1, 7]. Justify your answer.

(a) $h(1) = \int_1^1 f(t)dt = 0$



(b) $h'(x) = f(x) \quad \Rightarrow \quad h'(4) = f(4) = 2$

(c) $h''(x) = f'(x) \quad \Rightarrow \quad f'(x) = 0 \quad at \quad x = 3 \; or \; 6$

$h''(x)$: + on (1, 3), − on (3, 6), + on (6, 7)

1 3 6 7

$\boxed{\text{The graph of } h \text{ is concave upward on (1,3) or (6,7).}}$

(d) $h'(x) = f(x) \qquad \begin{cases} f(x) = 0 & at \quad x = 5 \\ f(x) > 0 & on \quad (1,5) \Rightarrow h(x) \; increases \; on \; (1,5) \\ f(x) < 0 & on \quad (5,7) \Rightarrow h(x) \; decreases \; on \; (5,7) \end{cases}$

$h'(x)$: + on (1, 5), − on (5, 7)

$h(x)$: 1 incr 5 decr 7

Candidates for absolute min are $h(1)$ and $h(7)$.

$h(1) = \int_1^1 f(t)\,dt = 0$

$h(7) = \int_1^7 f(t)\,dt > 0$ because the area above the x-axis is greater than the area below the axis,

or $\int_1^5 f(t)\,dt \; > \; \int_7^5 f(t)\,dt$

$\boxed{h \text{ has an absolute min at x=1}}$

1. Two particles move in the xy-plane. For time $t \geq 0$, the position of particle A is given by $x = t - 2$ and $y = (t-2)^2$, and the position of particle B is given by $x = \frac{3t}{2} - 4$ and $y = \frac{3t}{2} - 2$.
   (a) Find the velocity vector for each particle at time t = 3.
   (b) Set up an integral expression that gives the distance traveled by particle A from t = 0 to t = 3. Do not evaluate.
   (c) Determine the exact time at which the particles collide; that is, when the particles are at the same point at the same time. Justify your answer.
   (d) In the viewing window provided below, sketch the paths of particles A and B from t=0 until they collide. Indicate the direction of each particle along its path.

(a) A: $(t-2, (t-2)^2)$ B: $(\frac{3t}{2} - 4, \frac{3t}{2} - 2)$

$\mathbf{V}_A = (1, 2(t-2))$ $\mathbf{V}_B = (\frac{3}{2}, \frac{3}{2})$

$\boxed{\mathbf{V}_A\big|_{t=3} = (1,2) \quad , \quad \mathbf{V}_B\big|_{t=3} = (\frac{3}{2}, \frac{3}{2})}$

(b) $\boxed{S = \int_0^3 \sqrt{1 + 4(t-2)^2}\, dt}$

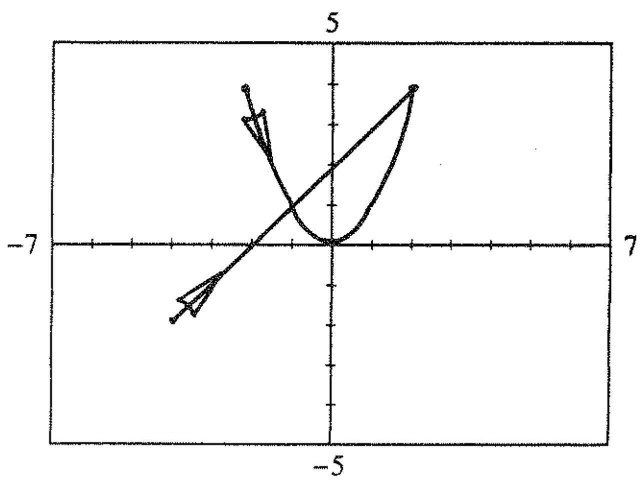
(c) $x_A = x_B \Rightarrow t - 2 = \frac{3t}{2} - 4 \Rightarrow t = 4$

When $t = 4$, $y_A = (4-2)^2 = 4$ *and* $y_B = \frac{3 \cdot 4}{2} - 2 = 4$; $t = 4 \Rightarrow x_A = x_B$ *and* $y_A = y_B$.

$\boxed{\text{Particles A and B collide when t = 4.}}$

(d) path of A: $y = x^2$
path of B: $y = x + 2$

(Note: A and B arrive at (–1, 1) at different times).



4. Let f be a function that has derivatives of all orders for all real numbers.
Assume $f(1)=3$, $f'(1)=-2$, $f''(1)=2$, and $f'''(1)=4$

(a) Write the second-degree Taylor polynomial for f about x = 1 and use it to approximate $f(0.7)$.

(b) Write the third-degree Taylor polynomial for f about x = 1 and use it to approximate $f(1.2)$.

(c) Write the second-degree Taylor polynomial for f', the derivative of f, about x = 1 and use it to approximate $f'(1.2)$.

(a) $f(x)=f(a)+f'(a)(x-a)+\dfrac{f''(a)}{2!}(x-a)^2+\dfrac{f'''(a)}{3!}(x-a)^3+\ \ldots$

$f(x)\approx f(1)+f'(1)(x-1)+\dfrac{f''(1)}{2!}(x-1)^2$

$f(x)\approx 3-2(x-1)+\frac{2}{2}(x-1)^2$

$f(0.7)\approx 3+0.6+0.09=3.69$

$\boxed{f(0.7)\approx 3.69}$

(b) $f(x)\approx f(1)+f'(1)(x-1)+\dfrac{f''(1)}{2!}(x-1)^2+\dfrac{f'''(1)}{3!}(x-1)^3$

$f(x)\approx 3-2(x-1)+(x-1)^2+\frac{4}{3!}(x-1)^3$

$f(1.2)\approx 3-0.4+0.04+\frac{2}{3}(0.008)=2.645\overline{3}$

$\boxed{f(1.2)\approx 2.645}$

(c) $f'(x)\approx -2+2(x-1)+2(x-1)^2$

$f'(1.2)\approx -2+2(.2)+2(.2)^2$

$\boxed{f'(1.2)\approx -1.52}$

1995 – BC5

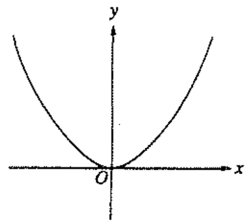


Figure 1
$y = f(x)$

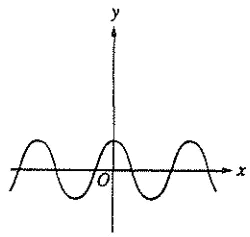


Figure 2
$y = g(x)$

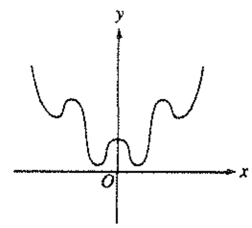
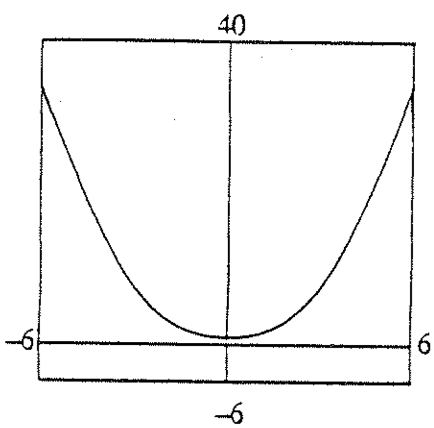


Figure 3

5. Let $f(x) = x^2$, $g(x) = \cos x$, and $h(x) = x^2 + \cos x$. From the graphs of f and g shown above in Figure 1 and Figure 2, one might think that the graph of h should look like the graph in Figure 3.

   (a) Sketch the actual graph of h in the viewing window provided below.
   (b) Use $h''(x)$ to explain why the graph of h does not look like the graph in Figure 3.
   (c) Prove that the graph of $y = x^2 + \cos(kx)$ has either no points of inflection or infinitely many points of inflection, depending on the value of the constant k.

(a)



(b) $h(x) = x^2 + \cos x \quad \Rightarrow \quad h'(x) = 2x - \sin x$
$h''(x) = 2 - \cos x \geq 1 \quad \forall x$. Therefore, the graph of h is concave upward $\forall x$ and has no points of inflection. Figure 3 has inflection points.

(c) $y = x^2 + \cos kx \quad \Rightarrow \quad \frac{dy}{dx} = 2x - k(\sin kx) \quad \Rightarrow \quad \frac{d^2y}{dx^2} = 2 - k^2(\cos kx) = 0 \quad \textit{if} \quad \cos(kx) = \frac{2}{k^2}$

i) $|k| > \sqrt{2} \quad \Rightarrow$ there are an infinite number of inflection points because $\frac{d^2y}{dx^2} = 0$ and changes sign at an infinite number of points.

ii) $|k| = \sqrt{2} \quad \Rightarrow$ there are no inflection points because $\frac{d^2y}{dx^2} = 0$ at some points but does not change sign.

iii) $|k| < \sqrt{2} \quad \Rightarrow$ there are no inflection points because $\frac{d^2y}{dx^2}$ is always positive.

1995 – BC6

6. Let f be a function whose domain is the closed interval [0, 5]. The graph of f is shown below.

   Let $h(x) = \int_0^{\frac{x}{2}+3} f(t)\,dt$.

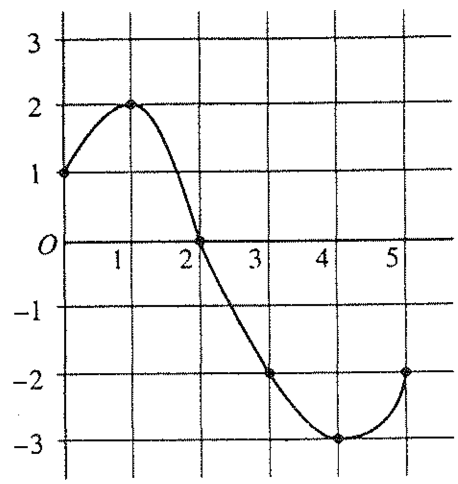
   (a) Find the domain of h.

   (b) Find $h'(2)$.

   (c) At what x is $h(x)$ a minimum? Show the analysis that leads to your conclusion.

(a) $D_f = [0,5] \Rightarrow 0 \le \frac{x}{2}+3 \le 5 \Rightarrow -6 \le x \le 4$

$\boxed{D_h = [-6,4] \quad or \quad -6 \le x \le 4}$



Graph of f

(b) $h'(x) = \frac{1}{2} f(\frac{x}{2}+3)$

$h'(2) = \frac{1}{2} f(\frac{2}{2}+3) = \frac{1}{2} f(4) = \frac{-3}{2}$

$\boxed{h'(2) = \frac{-3}{2}}$

(c) $h'(-2) = \frac{1}{2} f(\frac{-2}{2}+3) = \frac{1}{2} f(2) = 0$

$h'(x)$: on $[-6, -2]$: $+$; on $[-2, 4]$: $-$

$h(x)$: incr, decr

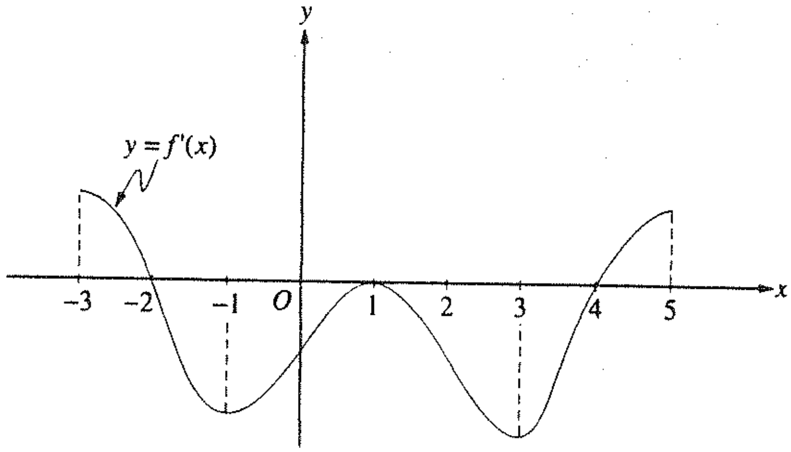
Candidates for absolute min are $h(-6)$ *and* $h(4)$.

$h(-6) = \int_0^{\frac{-6}{2}+3} f(t)\,dt = \int_0^0 f(t)\,dt = 0$

$h(4) = \int_0^{\frac{4}{2}+3} f(t)\,dt = \int_0^5 f(t)\,dt < 0$ from graph, or $\int_0^2 f(t)\,dt < \int_5^2 f(t)\,dt$

$\boxed{h \text{ has an absolute min at } x = 4.}$

1996: AB-1



Note: This is the graph of the derivative of f, not the graph of f.

The figure above shows the graph of f', the derivative of a function f. The domain of f is the set of all real numbers x such that $-3 < x < 5$.

(a) For what values of x does f have a relative maximum? Why?

(b) For what values of x does f have a relative minimum? Why?

(c) On what intervals is the graph of f concave upward? Use f' to justify your answer.

(d) Suppose that $f(1) = 0$. In the xy-plane provided, draw a sketch that shows the general shape of 'the graph of the function f on the open interval $0 < x < 2$.

Note: The axes for this graph are provided in the pink booklet only.

(a) f has a relative maximum at $x = -2$

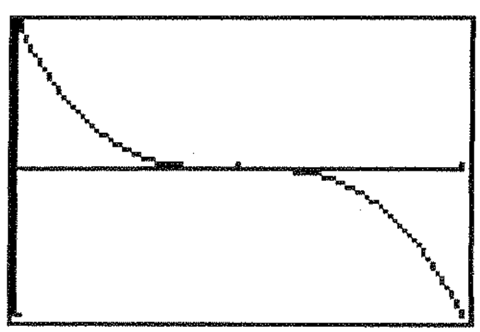
because $f'(-2) = 0$ and $f'(x) > 0$ on $(-3,-2)$ and $f'(x) < 0$ on $(-2,-1)$. ☺

(b) f has a relative minimum at $x = 4$

because $f'(4) = 0$ and $f'(x) < 0$ on $(3,4)$ and $f'(x) > 0$ on $(4,5)$. ☺

(c) $f'(x)$ increasing $\Longrightarrow f''(x) > 0 \Longrightarrow$ the graph of f is concave upward on $(-1,1)$ and $(3,5)$. ☺

(d)



Let R be the region in the first quadrant under the graph of $y = \frac{1}{\sqrt{x}}$ for $4 \le x \le 9$.

(a) Find the area of R.

(b) If the line $x = k$ divides the region R into two regions of equal area, what is the value of k ?

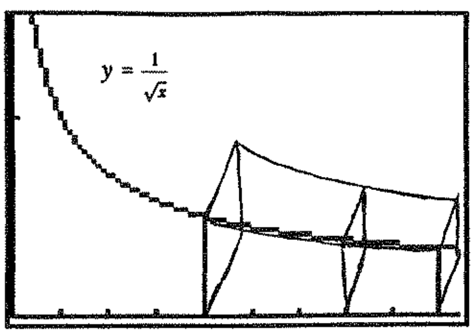
(c) Find the volume of the solid whose base is the region R and whose cross sections cut by planes perpendicular to the x-axis are squares.

(a) $A = \int_4^9 x^{-\frac{1}{2}}\,dx = 2x^{\frac{1}{2}}\Big|_4^9 = 2\left(9^{\frac{1}{2}} - 4^{\frac{1}{2}}\right) = 2(3-2) = 2$ ☺

(b) $\int_4^k x^{-\frac{1}{2}}\,dx = \int_k^9 x^{-\frac{1}{2}}\,dx \Longrightarrow 2x^{\frac{1}{2}}\Big|_4^k = 2x^{\frac{1}{2}}\Big|_k^9$

$\Longrightarrow \sqrt{k} - 2 = 3 - \sqrt{k} \Longrightarrow 2\sqrt{k} = 5 \Longrightarrow k = \frac{25}{4}$ ☺

(c)



$$A = \int_4^9 \left(\frac{1}{\sqrt{x}}\right)^2 dx = \int_4^9 \frac{1}{x}\,dx = \ln x \Big|_4^9 = \ln 9 - \ln 4 \ (\approx .811)$$ ☺

The rate of consumption of cola in the United States is given by $S(t) = Ce^{kt}$, where S is measured in billions of gallons per year and t is measured in years from the beginning of 1980.

(a) The consumption rate doubles every 5 years and the consumption rate at the beginning of 1980 was 6 billion gallons per year. Find C and k.

(b) Find the average rate of consumption of cola over the 10-year time period beginning January 1, 1983. Indicate units of measure.

(c) Use the trapezoidal rule with four equal subdivisions to estimate $\int_5^7 S(t)\,dt$.

(d) Using correct units, explain the meaning of $\int_5^7 S(t)\,dt$ in terms of cola consumption.

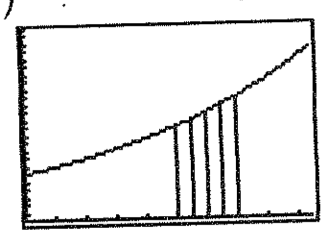
(a) $S = 6$ when $t = 0 \Longrightarrow C = 6$ ☺ $\Longrightarrow S(t) = 6e^{kt}$

$S = 12$ when $t = 5 \Longrightarrow 12 = 6e^{5k} \Longrightarrow 2 = e^{5k} \Longrightarrow 5k = \ln 2 \Longrightarrow k = \frac{1}{5}\ln 2$ ☺

(b) $S(t) = 6\cdot 2^{\frac{1}{5}t}$

$$S_{\text{avg}}(t) = \frac{1}{13-3}\int_3^{13} 6\cdot 2^{\frac{1}{5}t}\,dt = \frac{6}{10}\int_3^{13} 2^{\frac{1}{5}t}\,dt \quad \left(u = \tfrac{1}{5}t,\ du = \tfrac{1}{5}dt,\ 5\,du = dt\right)$$

$$= \frac{6}{10}\cdot 5\int_{\frac{3}{5}}^{\frac{13}{5}} 2^u\,du = \frac{3}{\ln 2}2^u\Big|_{\frac{3}{5}}^{\frac{13}{5}} = \frac{3}{\ln 2}\left(2^{\frac{13}{5}} - 2^{\frac{3}{5}}\right) \approx 19.680 \text{ billion gallons/year}$$ ☺



$y = 6(2)^{\frac{t}{5}}$

(c) $T = \Delta x\left(\frac{1}{2}y_0 + y_1 + y_2 + \cdots + y_{n-1} + \frac{1}{2}y_n\right)$

$$\int_5^7 6\cdot 2^{\frac{1}{5}t}\,dt \approx .5\left(\frac{1}{2}\cdot 6\cdot 2^{\frac{5}{5}} + 6\cdot 2^{\frac{5.5}{5}} + 6\cdot 2^{\frac{6}{5}} + 6\cdot 2^{\frac{6.5}{5}} + \frac{1}{2}\cdot 6\cdot 2^{\frac{7}{5}}\right) \approx 27.668$$ ☺

(d) $\int_5^7 6\cdot 2^{\frac{1}{5}t}\,dt$ = the amount, in billions of gallons, of cola consumed in the *two year* period from 1/1/85 to 1/1/87. ☺

1996 AB-4 BC-4 This problem deals with functions defined by $f(x) = x + b \sin x$, where b is a positive constant and $-2\pi \le x \le 2\pi$.

(a) Sketch the graphs of two of these functions, $y = x + \sin x$ and $y = x + 3 \sin x$, as indicated below.

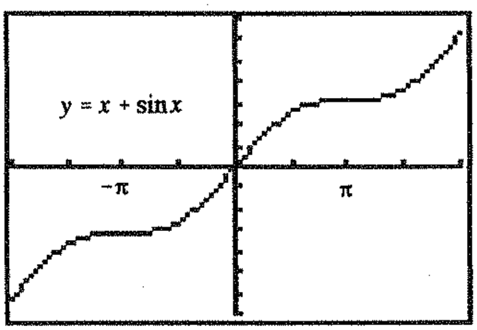
Note: The axes for these two graphs are provided in the pink test booklet only.

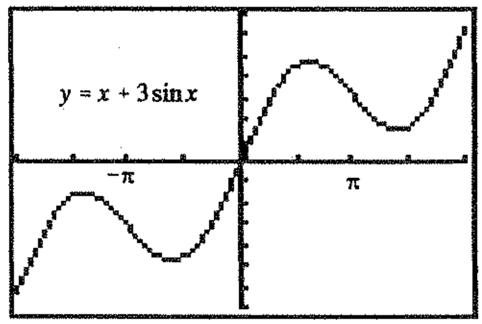
(b) Find the x-coordinates of all points, $-2\pi \le x \le 2\pi$, where the line $y = x + b$ is tangent to the graph of $f(x) = x + b \sin x$.

(c) Are the points of tangency described in part (b) relative maximum points of f? Why?

(d) For all values of $b > 0$, show that all inflection points of the graph of f lie on the line $y = x$.

(a)





(b) $y = x + b \Longrightarrow y' = 1$ and $f(x) = x + b\sin x \Longrightarrow f'(x) = 1 + b\cos x$

At the point of tangency, the y-coordinates are equal and the derivatives are equal.

$x + b = x + b\sin x \Longrightarrow \sin x = 1 \Longrightarrow x = \dfrac{\pi}{2}$ or $-\dfrac{3\pi}{2}$

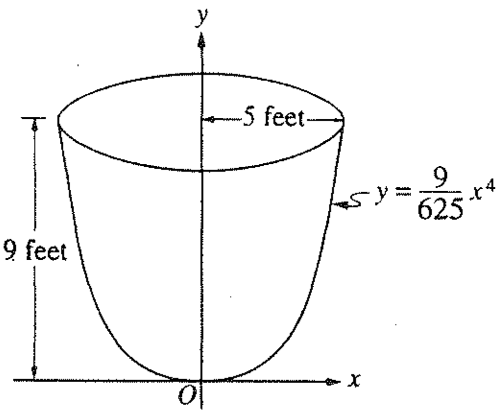
$1 = 1 + b\cos x \Longrightarrow \cos x = 0 \Longrightarrow x = \pm\dfrac{\pi}{2}$ or $x = \pm\dfrac{3\pi}{2}$

The graph of the line $y = x + b$ is tangent to the graph of $f(x)$ at $x = \dfrac{\pi}{2}$ and $x = -\dfrac{3\pi}{2}$. ☺

(c) No, the points of tangency are *not* relative maximum points of f because the slope (derivative) of f at the points of tangency is equal to one, but at relative maximum points the derivative must equal zero or be nonexistent. ☺

(d) $f(x) = x + b\sin x \Longrightarrow f'(x) = 1 + b\cos x \Longrightarrow f''(x) = -b\sin x$

At all inflection points of f, $f''(x) = 0$ which implies that $\sin x = 0$ (since $b > 0$) which implies that $f(x) = x$. Therefore, inflection points lie on the line $y = x$. ☺



An oil storage tank has the shape shown above, obtained by revolving the curve $y = \frac{9}{625}x^4$ from $x = 0$ to $x = 5$ about the y-axis, where x and y are measured in feet. Oil flows into the tank at the constant rate of 8 cubic feet per minute.

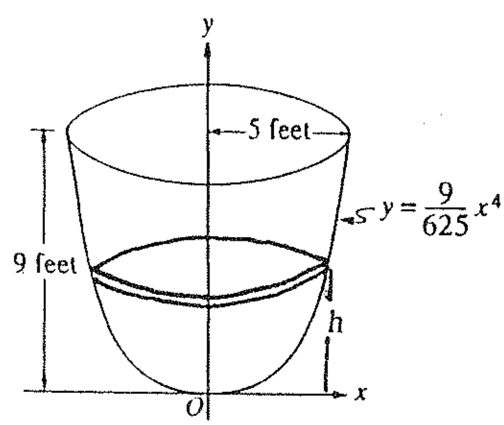
(a) Find the volume of the tank. Indicate units of measure.

(b) To the nearest minute, how long would it take to fill the tank if the tank was empty initially?

(c) Let h be the depth, in feet, of oil in the tank. How fast is the depth of the oil in the tank increasing when $h = 4$? Indicate units of measure.

(a) Horizontal disks: $V = \pi \int x^2 dy = \frac{25\pi}{3}\int_0^9 y^{\frac{1}{2}} dy = \frac{50\pi}{9} y^{\frac{3}{2}}\Big|_0^9 = \frac{50\pi}{9}\left(9^{\frac{3}{2}} - 0^{\frac{3}{2}}\right) = 150\pi \text{ ft}^3$ ☺

(b) $t = \dfrac{150\pi \text{ ft}^3}{8 \text{ ft}^3/\text{min}} \approx 58.905 \approx 59 \text{ min}$ ☺

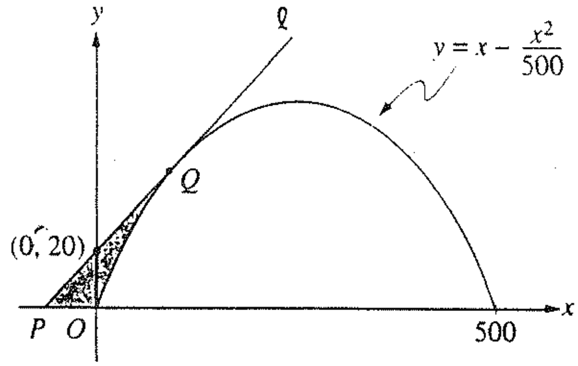


(c) $\dfrac{dV}{dt} = 8 \text{ ft}^3/\text{min}$; to evaluate $\left.\dfrac{dh}{dt}\right|_{h=4}$

$$V = \frac{50\pi}{9}h^{\frac{3}{2}} \Longrightarrow \frac{dV}{dt} = \frac{25\pi}{3}h^{\frac{1}{2}}\frac{dh}{dt}$$

$$\Longrightarrow 8 = \frac{25\pi}{3}\cdot 4^{\frac{1}{2}}\frac{dh}{dt} \Longrightarrow \frac{dh}{dt} = \frac{12}{25\pi}\ (\approx .153) \text{ ft/min}$$ ☺

1996: AB-6



Line ℓ is tangent to the graph of $y = x - \frac{x^2}{500}$ at the point Q, as shown in the figure above.

(a) Find the x-coordinate of point Q.

(b) Write an equation for line ℓ.

(c) Suppose the graph of $y = x - \frac{x^2}{500}$ shown in the figure, where x and y are measured in feet, represents a hill. There is a 50-foot tree growing vertically at the top of the hill. Does a spotlight at point P directed along line ℓ shine on any part of the tree? Show the work that leads to your conclusion.

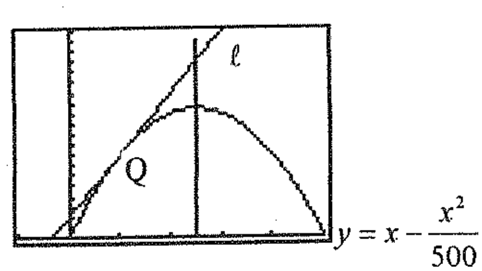
(a) $y = x - \frac{1}{500}x^2 \implies y' = 1 - \frac{1}{250}x = m$ and the equation of line ℓ is $y = mx + 20$.

On line ℓ at $Q(x_0, y_0)$: $y_0 = x_0 - \frac{1}{500}x_0^2 = (1 - \frac{1}{250}x_0)x_0 + 20$

$\implies x_0 - \frac{1}{500}x_0^2 = x_0 - \frac{1}{250}x_0^2 + 20 \implies \frac{1}{500}x_0^2 = 20 \implies x_0^2 = 10000 \implies x_0 = 100$ ☺

(b) $y'(100) = 1 - \frac{1}{250}\cdot 100 = \frac{3}{5} \implies$ the equation of line ℓ is $y = \frac{3}{5}x + 20$. ☺

(c) $y'(250) = 0$ and $y''(250) < 0 \implies$ a relative maximum at $x = 250$, the x-coordinate of the top of the hill. Therefore, the height of the hill is $y(250) = 125$ ft. and the top of the tree is 175 ft. above the x-axis. On the line, when $x = 250$, $y = 170$. Hence, the spotlight hits the tree 5 ft. below its top. ☺



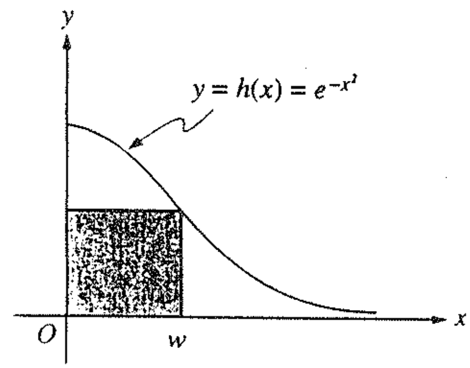
1996 AB6

1996: BC-1

Consider the graph of the function h given by $h(x) = e^{-x^2}$ for $0 \le x < \infty$.

(a) Let R be the unbounded region in the first quadrant below the graph of h. Find the volume of the solid generated when R is revolved about the y-axis.

(b) Let $A(w)$ be the area of the shaded rectangle shown in the figure to the right. Show that $A(w)$ has its maximum value when w is the x-coordinate of the point of inflection of the graph of h.



(a) $V = 2\pi\int_0^\infty xy\,dx = 2\pi\int_0^\infty xe^{-x^2}dx$ (cylindrical shells)

$= 2\pi \lim_{b\to\infty}\int_0^b xe^{-x^2}dx \quad (u = -x^2,\ du = -2x\,dx,\ -\frac{1}{2}du = x\,dx)$

$= -\pi \lim_{b\to\infty}\int_0^{-b^2} e^u du = -\pi \lim_{b\to\infty}\left(e^u\Big|_0^{-b^2}\right) = -\pi \lim_{b\to\infty}\left(e^{-b^2} - e^0\right) = \pi$ ☺

(b) $A(w) = we^{-w^2} \Longrightarrow \frac{dA}{dw} = A'(w) = -2w^2e^{-w^2} + e^{-w^2} = e^{-w^2}(-2w^2 + 1)$

$e^{-w^2}(-2w^2+1) = 0 \Longrightarrow -2w^2 + 1 = 0 \Longrightarrow w = \sqrt{\frac{1}{2}}$

$A'(\sqrt{\frac{1}{2}}) = 0$, $A'(w) > 0$ on $(0, \sqrt{\frac{1}{2}})$ and $A'(w) < 0$ on $(\sqrt{\frac{1}{2}}, \infty)$

$\Longrightarrow$ A has an absolute maximum at $w = \sqrt{\frac{1}{2}}$, $w > 0$ ★

$h'(x) = -2xe^{-x^2} \Longrightarrow h''(x) = -2(-2x^2e^{-x^2} + e^{-x^2} = -2e^{-x^2}(-2x^2 + 1)$

$h''(\sqrt{\frac{1}{2}}) = 0$, $h''(x) < 0$ on $(0, \sqrt{\frac{1}{2}})$ and $h''(x) > 0$ on $(\sqrt{\frac{1}{2}}, \infty)$

$\Longrightarrow$ h has a point of inflection at $x = \sqrt{\frac{1}{2}}$ ★ ☺

1996: BC-2

The Maclaurin series for $f(x)$ is given by $1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \ldots + \frac{x^n}{(n+1)!} + \ldots$

(a) Find $f'(0)$ and $f^{(17)}(0)$.

(b) For what values of x does the given series converge? Show your reasoning.

(c) Let $g(x) = xf(x)$. Write the Maclaurin series for $g(x)$, showing the first three nonzero terms and the general term.

(d) Write $g(x)$ in terms of a familiar function without using series. Then, write $f(x)$ in terms of the same familiar function.

(a)

$$f'(x) = \frac{1}{2!} + \frac{2x}{3!} + \frac{3x^2}{4!} + \cdots + \frac{nx^{n-1}}{(n+1)!} + \cdots \Longrightarrow f'(0) = \frac{1}{2!} = \frac{1}{2} \quad ☺$$

$$f''(x) = \frac{2!}{3!} + \frac{3 \cdot 2x}{4!} + \cdots + \frac{n(n-1)x^{n-2}}{(n+1)!} \cdots \Longrightarrow f''(0) = \frac{2!}{3!} = \frac{1}{3}$$

$$f^{(17)}(x) = \frac{17!}{18!} + \frac{18 \cdot 17 \cdots 3 \cdot 2x}{19!} + \cdots + \frac{n(n-1)\cdots(n-16)x^{n-17}}{(n+1)!} + \cdots \Longrightarrow f^{(17)}(0) = \frac{17!}{18!} = \frac{1}{18} \quad ☺$$

(b) Using the ratio test for absolute convergence:

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+2)!}}{\frac{x^n}{(n+1)!}}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+2)!}\cdot\frac{(n+1)!}{x^n}\right| = \lim_{n\to\infty}\left|\frac{x}{n+2}\right| = 0, \ \forall x \in \mathbb{R}$$

Therefore, the series converges for all real numbers. ☺

(c) $g(x) = xf(x) = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n+1}}{(n+1)!} + \cdots$ ☺

(d) $g(x) = e^x - 1$ ☺

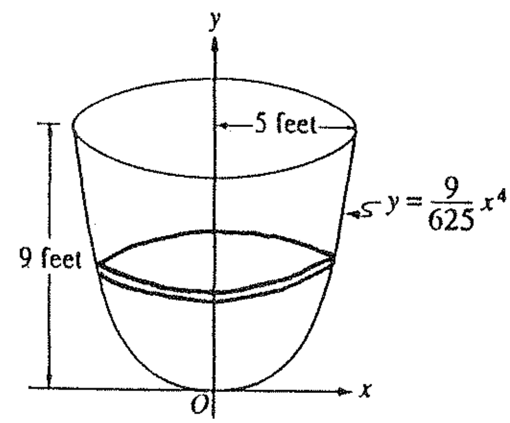
$$f(x) = \frac{g(x)}{x} = \begin{cases} \frac{e^x - 1}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases} \quad ☺$$

An oil storage tank has the shape shown above, obtained by revolving the curve $y = \frac{9}{625}x^4$ from $x = 0$ to $x = 5$ about the y-axis, where x and y are measured in feet. Oil weighing 50 pounds per cubic foot flowed into an initially empty tank at a constant rate of 8 cubic feet per minute. When the depth of the oil reached 6 feet, the flow stopped.

(a) Let h be the depth, in feet, of oil in the tank. How fast was the depth of the oil in the tank increasing when $h = 4$? Indicate units of measure.

(b) Find, to the nearest foot-pound, the amount of work required to empty the tank by pumping all of the oil back to the top of the tank.

(a) Horizontal disks: $V = \pi\int x^2 dy = \frac{25\pi}{3}\int_0^h y^{\frac{1}{2}}dy = \frac{50\pi}{9}h^{\frac{3}{2}}$



$\frac{dV}{dt} = 8\text{ ft}^3/\text{min}$; to evaluate $\left.\frac{dh}{dt}\right|_{h=4}$

$V = \frac{50\pi}{9}h^{\frac{3}{2}} \Longrightarrow \frac{dV}{dt} = \frac{25\pi}{3}h^{\frac{1}{2}}\frac{dh}{dt}$

$\Longrightarrow 8 = \frac{25\pi}{3}\cdot 4^{\frac{1}{2}}\frac{dh}{dt} \Longrightarrow \frac{dh}{dt} = \frac{12}{25\pi}$ ($\approx .153$) ft/min ☺

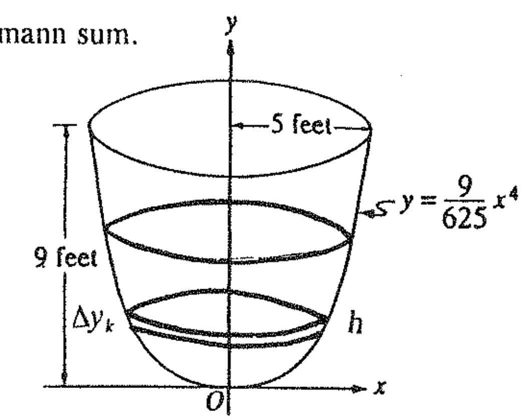
(b) F_k("k^{th} layer") = (volume)(density) = $(\pi r^2\Delta y_k\text{ ft}^3)(50\text{ lb/ft}^3) = \pi\frac{25}{3}y_k^{\frac{1}{2}}\Delta y_k(50) = \frac{1250}{3}\pi y_k^{\frac{1}{2}}\Delta y_k$ lb

$W = \Sigma W_k = \Sigma F_k d_k = \sum(\frac{1250}{3}\pi y_k^{\frac{1}{2}}\Delta y_k\text{ lb})(9 - y_k)\text{ft}$, a Riemann sum.

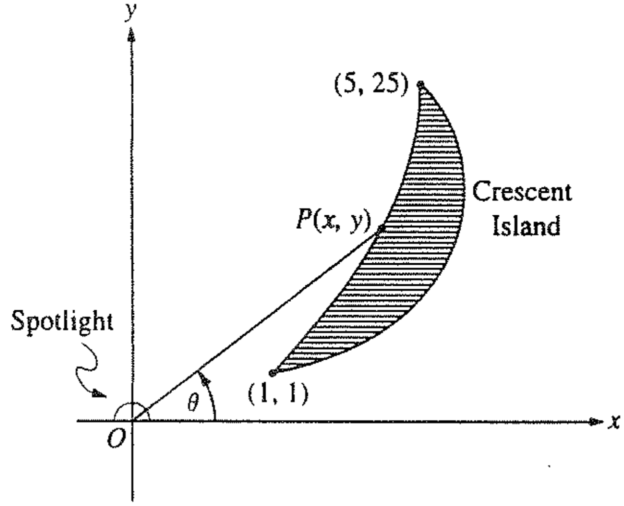
$W = \frac{1250}{3}\pi\int_0^6 y^{\frac{1}{2}}(9-y)\,dy = \frac{1250}{3}\pi\int_0^6\left(9y^{\frac{1}{2}} - y^{\frac{3}{2}}\right)dy$

$= \frac{1250}{3}\pi\left(9\cdot\frac{2}{3}y^{\frac{3}{2}} - \frac{2}{5}y^{\frac{5}{2}}\right)\Big|_0^6 = \frac{1250}{3}\pi\left(6\cdot 6^{\frac{3}{2}} - \frac{2}{5}\cdot 6^{\frac{5}{2}}\right)$

$= 250\pi\cdot 6^{\frac{5}{2}} \approx 69257.691 \approx 69258$ ft-lb ☺



1996: BC-6



Note: Figure not drawn to scale.

The figure above shows a spotlight shining on point $P(x, y)$ on the shoreline of Crescent Island. The spotlight is located at the origin and is rotating. The portion of the shoreline on which the spotlight shines is in the shape of the parabola $y = x^2$ from the point (1, 1) to the point (5, 25). Let θ be the angle between the beam of light and the positive x-axis.

(a) For what values of θ between 0 and 2π does the spotlight shine on the shoreline?

(b) Find the x- and y-coordinates of point P in terms of $\tan\theta$.

(c) If the spotlight is rotating at the rate of one revolution per minute, how fast is the point P traveling along the shoreline at the instant it is at the point (3, 9) ?

(a) The spotlight shines on the shoreline for θ such that

$\tan^{-1}\frac{1}{1} \le \theta \le \tan^{-1}\frac{25}{5} \Longrightarrow \frac{\pi}{4} \le \theta \le \tan^{-1}5 \Longrightarrow .785 \le \theta \le 1.373$ ☺

(b) $\tan\theta = \dfrac{y}{x} = \dfrac{x^2}{x} \Longrightarrow x = \tan\theta$ and $y = x^2 = \tan^2\theta$ ☺

(c) $\vec{r} = \langle \tan\theta, \tan^2\theta \rangle \Longrightarrow \vec{v} = \langle \sec^2\theta\dfrac{d\theta}{dt}, 2\tan\theta\sec^2\theta\dfrac{d\theta}{dt} \rangle$

$\dfrac{d\theta}{dt} = 2\pi$ units/min, and at the point (3,9), $\tan\theta = 3$ and $\sec\theta = \sqrt{10}$ $\quad(\sec^2\theta = \tan^2\theta + 1)$

$$|\vec{v}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\sec^4\theta\left(\frac{d\theta}{dt}\right)^2 + 4\tan^2\theta\sec^4\theta\left(\frac{d\theta}{dt}\right)^2}$$

$$= \sqrt{100(4\pi^2) + 4(9)(100)(4\pi^2)} = 20\pi\sqrt{37} \approx 382.191 \text{ units/min}$$ ☺

1997: AB-1

A particle moves along the x-axis so that its velocity at any time $t \geq 0$ is given by $v(t) = 3t^2 - 2t - 1$. The position $x(t)$ is 5 for $t = 2$.

(a) Write a polynomial expression for the position of the particle at any time $t \geq 0$.

(b) For what values of t, $0 \leq t \leq 3$, is the particle's instantaneous velocity the same as its average velocity on the closed interval $[0, 3]$?

(c) Find the total distance traveled by the particle from time $t = 0$ until time $t = 3$.

(a) $x(t) = \int v(t)\,dt = t^3 - t^2 - t + C$

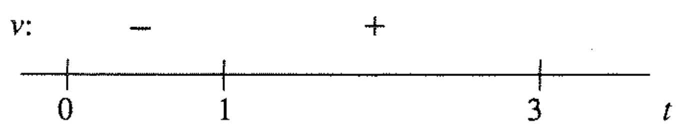
$x(2) = 5 \Longrightarrow C = 3$

$x(t) = t^3 - t^2 - t + 3$ ☺

(b) $v_{avg}(t) = \dfrac{x(3) - x(0)}{3 - 0} = \dfrac{18 - 3}{3} = 5$

$v(t) = 3t^2 - 2t - 1 = 5 \Longrightarrow 3t^2 - 2t - 6 = 0 \Longrightarrow$ since $t \geq 0$, $t = \dfrac{2 + \sqrt{76}}{6} \approx 1.786$ ☺

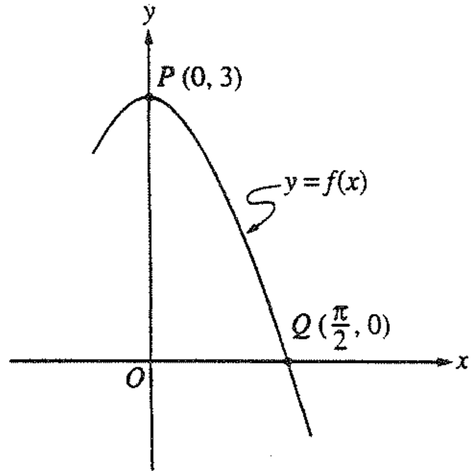
(c) $v(t) = 3t^2 - 2t - 1 = (3t + 1)(t - 1)$



$x(0) = 3,\ x(1) = 2,\ x(3) = 18 \Longrightarrow$ total distance $= (3 - 2) + (18 - 2) = 17$

OR $s(t) = \int_0^3 |v(t)|\,dt = \int_0^1 (1 + 2t - 3t^2)\,dt + \int_1^3 (3t^2 - 2t - 1)\,dt = \left[t + t^2 - t^3\right]_0^1 + \left[t^3 - t^2 - t\right]_1^3 = 17$

The total distance traveled on $t \in [0,3]$ is 17. ☺



Let f be the function given by $f(x) = 3\cos x$. As shown above, the graph of f crosses the y-axis at point P and the x-axis at point Q.

(a) Write an equation for the line passing through points P and Q.

(b) Write an equation for the line tangent to the graph of f at point Q. Show the analysis that leads to your equation.

(c) Find the x-coordinate of the point on the graph of f, between points P and Q, at which the line tangent to the graph of f is parallel to line PQ.

(d) Let R be the region in the first quadrant bounded by the graph of f and line segment PQ. Write an integral expression for the volume of the solid generated by revolving the region R about the x-axis. Do not evaluate.

(a) $m_{\overline{PQ}} = \dfrac{3-0}{0-\frac{\pi}{2}} = -\dfrac{6}{\pi} \Longrightarrow$ Equation of $\overleftrightarrow{PQ}$: $y - 0 = -\frac{6}{\pi}(x - \frac{\pi}{2})$ or $y = -\frac{6}{\pi}x + 3$ ☺

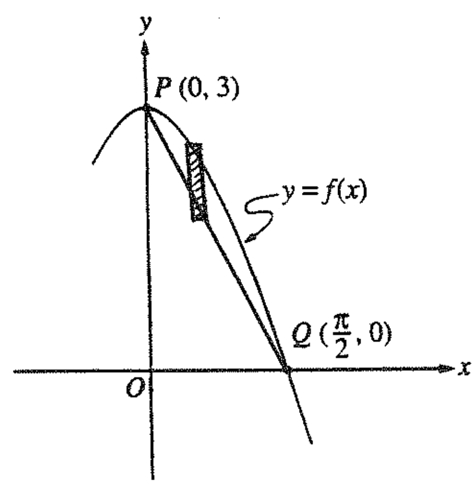
(b) $m_T(x) = f'(x) = -3\sin x \Longrightarrow m_T\left(\frac{\pi}{2}\right) = -3$

Equation of tangent to $y = f(x)$ at point Q: $y - 0 = -3(x - \frac{\pi}{2})$ or $y = -3x + \frac{3\pi}{2}$ ☺

(c) $m_T(x) = m_{\overline{PQ}} \Longrightarrow -3\sin x = -\frac{6}{\pi} \Longrightarrow \sin x = \frac{2}{\pi}$

$\Longrightarrow x = \sin^{-1}\left(\frac{2}{\pi}\right) \approx .690$ ☺

(d) $V = \pi \displaystyle\int_0^{\frac{\pi}{2}} \left[(3\cos x)^2 - \left(-\frac{6}{\pi}x + 3\right)^2\right] dx$ ☺



1997: AB-3

Let f be the function given by $f(x) = \sqrt{x - 3}$.

(a) On the axes provided below, sketch the graph of f and shade the region R enclosed by the graph of f, the x-axis, and the vertical line $x = 6$.

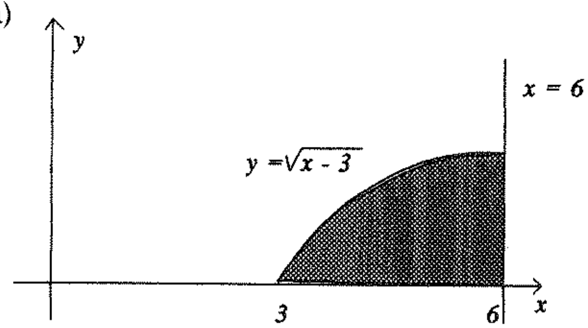
Note: The axes for this graph are provided in the pink test booklet only.

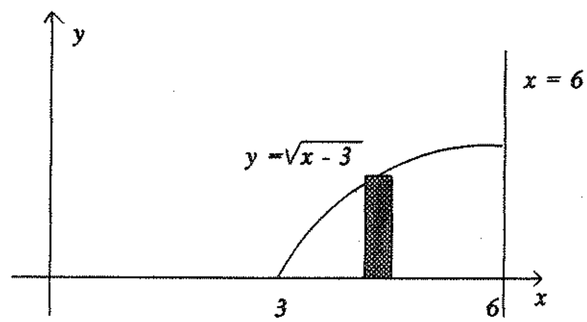
(b) Find the area of the region R described in part (a).

(c) Rather than using the line $x = 6$ as in part (a), consider the line $x = w$, where w can be any number greater than 3. Let $A(w)$ be the area of the region enclosed by the graph of f, the x-axis, and the vertical line $x = w$. Write an integral expression for $A(w)$.

(d) Let $A(w)$ be as described in part (c). Find the rate of change of A with respect to w when $w = 6$.

(a)





(b) $A = \int_3^6 (x-3)^{\frac{1}{2}}\,dx = \left[\frac{2}{3}(x-3)^{\frac{3}{2}}\right]_3^6 = \frac{2}{3}\cdot 3^{\frac{3}{2}} = 2\sqrt{3} \approx 3.464$ ☺

(c) $A(w) = \int_3^w (x-3)^{\frac{1}{2}}\,dx$ ☺

(d) By the Fundamental Theorem, $A'(w) = (w-3)^{\frac{1}{2}} \Longrightarrow A'(6) = \sqrt{3}$ ☺

1997: AB-4

Let f be the function given by $f(x) = x^3 - 6x^2 + p$, where p is an arbitrary constant.

(a) Write an expression for $f'(x)$ and use it to find the relative maximum and minimum values of f in terms of p. Show the analysis that leads to your conclusion.

(b) For what values of the constant p does f have 3 distinct real roots?

(c) Find the value of p such that the average value of f over the closed interval $[-1, 2]$ is 1.

(a) $f(x) = x^3 - 6x^2 + p \Longrightarrow f'(x) = 3x^2 - 12x = 3x(x-4)$

f' test: use number line

f':	$+$		$-$		$+$	
		0		4		x
function:	incr		decr		incr	

Relative maximum at $x = 0 \Longrightarrow f(0) = p$ is the relative maximum value ☺

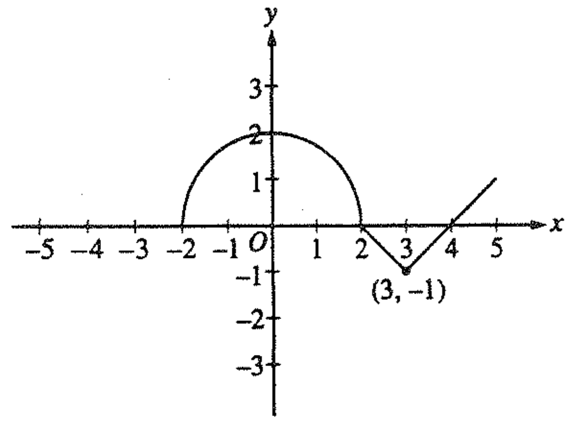
Relative minimum at $x = 4 \Longrightarrow f(4) = p - 32$ is the relative minimum value ☺

(b) $f(x)$ is a cubic polynomial function with a relative maximum and a relative minimum.
$\therefore$ $f(x)$ has three distinct zeros *iff* the relative maximum value is positive *and* the relative minimum value is negative. $p > 0$ and $p - 32 < 0 \Longrightarrow 0 < p < 32$ ☺

(c) $f_{avg}(x) = \frac{1}{3}\int_{-1}^{2} (x^3 - 6x^2 + p)\,dx = 1$

$\frac{1}{3}\left[\frac{1}{4}x^4 - 2x^3 + px\right]_{-1}^{2} = 1$

$(4 - 16 + 2p) - \left(\frac{1}{4} + 2 - p\right) = 3 \Longrightarrow p = 5.75$ ☺



The graph of a function f consists of a semicircle and two line segments as shown above. Let g be the function given by $g(x) = \int_0^x f(t)\,dt$.

(a) Find $g(3)$.

(b) Find all values of x on the open interval $(-2, 5)$ at which g has a relative maximum. Justify your answer.

(c) Write an equation for the line tangent to the graph of g at $x = 3$.

(d) Find the x-coordinate of each point of inflection of the graph of g on the open interval $(-2, 5)$. Justify your answer.

(a) $g(3) = \int_0^3 f(t)\,dt = \frac{1}{4}\pi r^2 - \frac{1}{2}bh = \frac{1}{4}\pi(2^2) - \frac{1}{2}(1)(1) = \pi - \frac{1}{2}$ ☺

(b) $g(x) = \int_0^x f(t)\,dt \Longrightarrow g'(x) = f(x)$

$g'(x)$:		+		−	+
x	-2		2	4	5
g:		incr		decr	incr

Since $g'(x)$ changes from positive to negative at $x = 2$, g has a relative maximum at $x = 2$ ☺

(c) $m = g'(3) = f(3) = -1$ and the point of tangency is $\left(3, \pi - \frac{1}{2}\right)$

$\therefore$ the equation of the tangent is $y - \left(\pi - \frac{1}{2}\right) = -1(x - 3)$ ☺

(d) Since g is differentiable on $(-2, 5)$, g is continuous on $[-2, 5]$.

$g'' = f'$:		+		−		+	
x	-2		0		3		5
g:		conc. up		conc. dn		conc. up	

g has points of inflection at $x = 0$ and $x = 3$ since $g''(x)$ changes sign at each of these values. ☺

1997: BC-1

During the time period from $t = 0$ to $t = 6$ seconds, a particle moves along the path given by $x(t) = 3\cos(\pi t)$ and $y(t) = 5\sin(\pi t)$.

(a) Find the position of the particle when $t = 2.5$.

(b) On the axes provided below, sketch the graph of the path of the particle from $t = 0$ to $t = 6$. Indicate the direction of the particle along its path.

Note: The axes for this graph are provided in the pink test booklet only.

(c) How many times does the particle pass through the point found in part (a) ?

(d) Find the velocity vector for the particle at any time t.

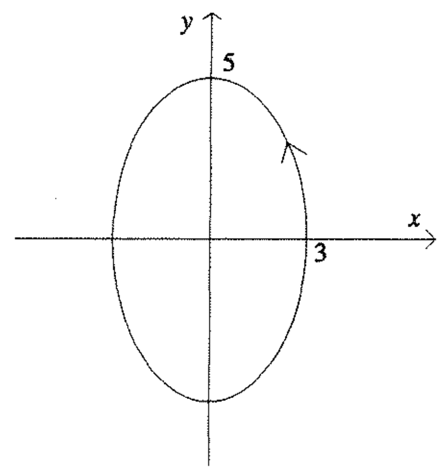
(e) Write and evaluate an integral expression, in terms of sine and cosine, that gives the distance the particle travels from time $t = 1.25$ to $t = 1.75$.

(a) $x(2.5) = 3\cos\frac{5\pi}{2} = 0$ and $y(2.5) = 5\sin\frac{5\pi}{2} = 5$.

At $t = 2.5$, the position of the particle is $(0,5)$. ☺

(b) $\frac{x}{3} = \cos\pi t$ and $\frac{y}{5} = \sin\pi t$

$$\frac{x^2}{9} + \frac{y^2}{25} = \cos^2\pi t + \sin^2\pi t = 1 \text{ (an ellipse)}$$



(c) The particle passes through $(0,5)$ three times. ☺

(at $t = .5,\ 2.5,\ 4.5$)

(d) $\vec{r} = \langle 3\cos\pi t, 5\sin\pi t\rangle \Longrightarrow \vec{v} = \langle -3\pi\sin\pi t, 5\pi\cos\pi t\rangle$ ☺

(e) $$s = \int_{t_1}^{t_2} |\vec{v}(t)|\,dt = \int_{1.25}^{1.75} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_{1.25}^{1.75} \sqrt{9\pi^2\sin^2\pi t + 25\pi^2\cos^2\pi t}\,dt \approx 5.392$$ ☺

Let $v(t)$ be the velocity, in feet per second, of a skydiver at time t seconds, $t \geq 0$. After her parachute opens, her velocity satisfies the differential equation $\frac{dv}{dt} = -2v - 32$, with initial condition $v(0) = -50$.

(a) Use separation of variables to find an expression for v in terms of t, where t is measured in seconds.

(b) Terminal velocity is defined as $\lim_{t\to\infty} v(t)$. Find the terminal velocity of the skydiver to the nearest foot per second.

(c) It is safe to land when her speed is 20 feet per second. At what time t does she reach this speed?

(a) $\frac{dv}{dt} = -2(v+16) \Longrightarrow \int \frac{1}{v+16}\,dv = \int -2\,dt \Longrightarrow \ln|v+16| = -2t + C_1$

$\Longrightarrow |v+16| = e^{-2t+C_1} = e^{-2t}e^{C_1} = C_2 e^{-2t} \Longrightarrow v + 16 = C_3 e^{-2t} \Longrightarrow v = C_3 e^{-2t} - 16$ ★

$v(0) = -50 \Longrightarrow C_3 = -34 \Longrightarrow v = -34e^{-2t} - 16$ ☺

(b) $\lim_{t\to\infty} v(t) = \lim_{t\to\infty}\left(-34e^{-2t} - 16\right) = -16$. The terminal velocity is -16 ft/sec ☺

(c) $v = -34e^{-2t} - 16$

$-20 = -34e^{-2t} - 16 \Longrightarrow e^{-2t} = \frac{2}{17} \Longrightarrow e^{2t} = \frac{17}{2} \Longrightarrow 2t = \ln\frac{17}{2} \Longrightarrow t = \frac{1}{2}\ln\frac{17}{2}$

She reaches a speed of 20 ft/sec at $t = \frac{1}{2}\ln\frac{17}{2} \approx 1.070$ seconds ☺

Let $P(x) = 7 - 3(x-4) + 5(x-4)^2 - 2(x-4)^3 + 6(x-4)^4$ be the fourth-degree Taylor polynomial for the function f about 4. Assume f has derivatives of all orders for all real numbers.

(a) Find $f(4)$ and $f'''(4)$.

(b) Write the second-degree Taylor polynomial for f' about 4 and use it to approximate $f'(4.3)$.

(c) Write the fourth-degree Taylor polynomial for $g(x) = \int_4^x f(t)\,dt$ about 4.

(d) Can $f(3)$ be determined from the information given? Justify your answer.

(a) $f(x) = f(4) + f'(4)(x-4) + \frac{f''(4)}{2!}(x-4)^2 + \frac{f'''(4)}{3!}(x-4)^3 + \frac{f^{(4)}(4)}{4!}(x-4)^4 + \cdots$

$P(x) = 7 - 3(x-4) + 5(x-4)^2 - 2(x-4)^3 + 6(x-4)^4$

$\therefore\ f(4) = 7$ and $\frac{f'''(4)}{3!} = -2 \Longrightarrow f'''(4) = -12$ ☺

(b) $f'(x) \approx P'(x) = -3 + 10(x-4) - 6(x-4)^2$ ☺

$f'(4.3) \approx -3 + 10(.3) - 6(.3)^2 = -.54$ ☺

(c) $g(x) \approx \int_4^x \left(7 - 3(t-4) + 5(t-4)^2 - 2(t-4)^3\right) dt$

$\approx 7t - \frac{3}{2}(t-4)^2 + \frac{5}{3}(t-4)^3 - \frac{1}{2}(t-4)^4 \Big]_4^x$

$\approx 7x - \frac{3}{2}(x-4)^2 + \frac{5}{3}(x-4)^3 - \frac{1}{2}(x-4)^4 - 28$

$g(x) \approx 7(x-4) - \frac{3}{2}(x-4)^2 + \frac{5}{3}(x-4)^3 - \frac{1}{2}(x-4)^4$ ☺

(d) NO. The information given provides values for $f(4)$, $f'(4)$, $f''(4)$, $f'''(4)$, and $f^{(4)}(4)$ only. ☺

Let R be the region enclosed by the graphs of $y = \ln(x^2+1)$ and $y = \cos x$.

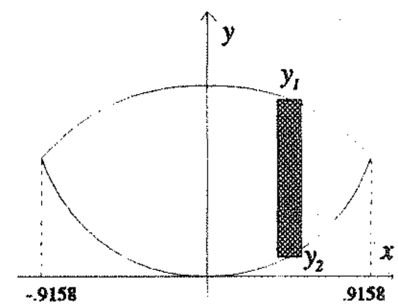
(a) Find the area of R.

(b) Write an expression involving one or more integrals that gives the length of the boundary of the region R. Do not evaluate.

(c) The base of a solid is the region R. Each cross section of the solid perpendicular to the x-axis is an equilateral triangle. Write an expression involving one or more integrals that gives the volume of the solid. Do not evaluate.

(a) Let $y_1 = \cos x$ and $y_2 = \ln(x^2+1)$ and use "solve" or "intersect" on calculator to determine points of intersection at $\pm.9158$

$$A = \int_{-.9158}^{.9158} (\cos x - \ln(x^2+1))\,dx \approx 1.168$$ ☺

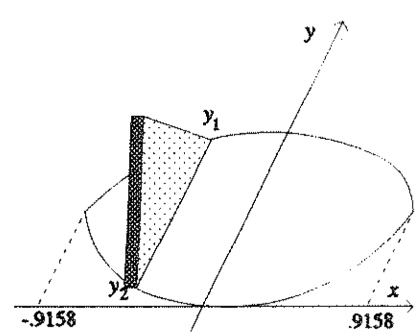


(b) $$s = \int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx, \quad \frac{dy_1}{dx} = -\sin x, \quad \frac{dy_2}{dx} = \frac{2x}{x^2+1}$$

$$s = s_1 + s_2 = \int_{-.9158}^{.9158} \left[\sqrt{1+\sin^2 x} + \sqrt{1+\left(\frac{2x}{x^2+1}\right)^2}\right] dx$$ ☺

(c) For an equilateral triangle, $A = \frac{s^2\sqrt{3}}{4}$ and the solid's cross section has $s = \cos x - \ln(x^2+1)$

$$V = \int_{x_1}^{x_2} A(x)\,dx = \frac{\sqrt{3}}{4}\int_{-.9158}^{.9158} [\cos x - \ln(x^2+1)]^2\,dx$$ ☺



1997: BC-4

Let $x = ky^2 + 2$, where $k > 0$.

(a) Show that for all $k > 0$, the point $\left(4, \sqrt{\frac{2}{k}}\right)$ is on the graph of $x = ky^2 + 2$.

(b) Show that for all $k > 0$, the tangent line to the graph of $x = ky^2 + 2$ at the point $\left(4, \sqrt{\frac{2}{k}}\right)$ passes through the origin.

(c) Let R be the region in the first quadrant bounded by the x-axis, the graph of $x = ky^2 + 2$, and the line $x = 4$. Write an integral expression for the area of the region R and show that this area decreases as k increases.

(a) $x = ky^2 + 2;\quad 4 \stackrel{?}{=} k\sqrt{\frac{2}{k}}^{\,2} + 2 = k\left(\frac{2}{k}\right) + 2 = 4$ Therefore, $\left(4, \sqrt{\frac{2}{k}}\right)$ is on the graph. ☺

(b) $\frac{dx}{dy} = 2ky \Longrightarrow \left.\frac{dy}{dx}\right|_{\left(4,\sqrt{\frac{2}{k}}\right)} = \left.\frac{1}{2ky}\right|_{\left(4,\sqrt{\frac{2}{k}}\right)} = \frac{1}{2k\sqrt{\frac{2}{k}}} = \frac{1}{2\sqrt{2k}}$

Equation of tangent at $\left(4, \sqrt{\frac{2}{k}}\right)$: $y - \sqrt{\frac{2}{k}} = \frac{1}{2\sqrt{2k}}(x - 4)$ ★

$0 - \sqrt{\frac{2}{k}} \stackrel{?}{=} \frac{1}{2\sqrt{2k}}(0 - 4) = \frac{-2}{\sqrt{2k}} = \frac{-\sqrt{2}}{\sqrt{k}} = -\sqrt{\frac{2}{k}}$ Therefore, the origin is on the tangent. ☺

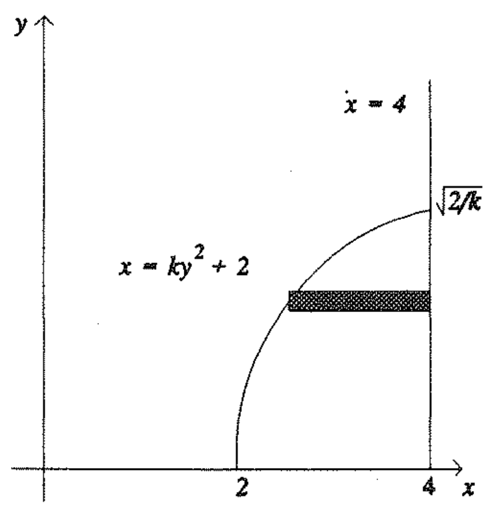
(c) $A = \int_0^{\sqrt{2/k}} \left[4 - (ky^2 + 2)\right] dy$ ☺

$= \int_0^{\sqrt{2/k}} (2 - ky^2)\, dy = \left[2y - \frac{1}{3}ky^3\right]_0^{\sqrt{2/k}}$

$= \left(\frac{2\sqrt{2}}{\sqrt{k}} - \frac{2\sqrt{2}}{3\sqrt{k}}\right) - 0 = \frac{4\sqrt{2}}{3\sqrt{k}}$ ★

$\frac{dA}{dk} = -\frac{2\sqrt{2}}{3}k^{-3/2} < 0.$

$\therefore$ the area decreases as k increases. ☺



1998: AB-1

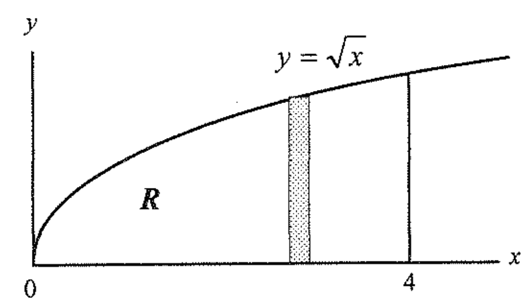
Let R be the region bounded by the x-axis, the graph of $y = \sqrt{x}$, and the line $x = 4$.

(a) Find the area of the region R.

(b) Find the value of h such that the vertical line $x = h$ divides the region R into two regions of equal area.

(c) Find the volume of the solid generated when R is revolved about the x-axis.

(d) The vertical line $x = k$ divides the region R into two regions such that when these two regions are revolved about the x-axis, they generate solids with equal volumes. Find the value of k.



(a) $A = \int_0^4 x^{\frac{1}{2}}dx = \frac{2}{3}x^{\frac{3}{2}}\Big|_0^4 = \frac{2}{3}(8-0) = \frac{16}{3}$ units^2 ☺

(b) $\int_0^h x^{\frac{1}{2}}dx = \frac{8}{3}$

$\frac{2}{3}x^{\frac{3}{2}}\Big|_0^h = \frac{8}{3}$

$\frac{2}{3}h^{\frac{3}{2}} = \frac{8}{3}$

$h^{\frac{3}{2}} = 4 \Longrightarrow h = \sqrt[3]{16} \approx 2.520$ ☺

(c) disks: $V = \pi\int_0^4 \sqrt{x}^2 dx = \pi\int_0^4 x\,dx = \frac{\pi}{2}x^2\Big|_0^4 = 8\pi$ units^3 ☺

(d) $\pi\int_0^k x\,dx = 4\pi \Longrightarrow \frac{\pi k^2}{2} = 4\pi \Longrightarrow k = \sqrt{8} \approx 2.828$ ☺

1998: AB-2; BC-2

Let f be the function given by $f(x) = 2xe^{2x}$.

(a) Find $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$.

(b) Find the absolute minimum value of f. Justify that your answer is an absolute minimum.

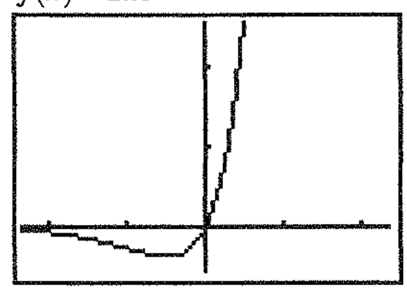
(c) What is the range of f?

(d) Consider the family of functions defined by $y = bxe^{bx}$, where b is a nonzero constant. Show that the absolute minimum value of bxe^{bx} is the same for all nonzero values of b.

(a) By L'Hospital's Rule (actually by John Bernoulli):

OR OBSERVE GRAPH

$f(x) = 2xe^{2x}$



$$\lim_{x\to-\infty} 2xe^{2x} = \lim_{x\to-\infty} \frac{2x}{e^{-2x}} = \lim_{x\to-\infty} \frac{2}{-2e^{-2x}} = 0. \quad ☺$$

$$\lim_{x\to\infty} 2xe^{2x} = \infty \quad ☺$$

window: $-2.35 \le x \le 2.35$
$-.55 \le y \le 2.55$

(b) $f'(x) = 2x(2e^{2x}) + 2e^{2x} = 2e^{2x}(2x+1)$

$f'(x) = 0 \Longrightarrow x = -\frac{1}{2}$

f': $-$ | $+$ (x)

f: decr $-\frac{1}{2}$ incr

By the first derivative test, $f(x)$ has its absolute minimum value at $x = -\frac{1}{2}$.

The absolute minimum value is $f\left(-\frac{1}{2}\right) = -\frac{1}{e} \approx -.368$ ☺

(c) Since $f(x)$ is continuous and $\lim_{x\to\infty} f(x) = \infty$, the range is $\left\{y \mid y \ge -\frac{1}{e}\right\}$ ☺.

(d) $y = bxe^{bx}$, $b \ne 0 \Longrightarrow y' = bxbe^{bx} + be^{bx} = be^{bx}(bx+1)$

$y' = 0 \Longrightarrow x = -\frac{1}{b}$

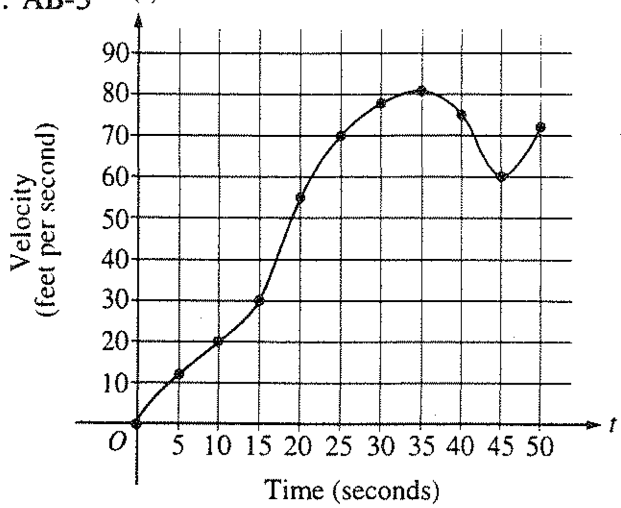
f': $-$ | $+$ (x)

f: decr $-\frac{1}{b}$ incr

For both $b > 0$ and $b < 0$, there is a relative *and* absolute minimum at $x = -\frac{1}{b}$, $\forall b \ne 0$.

The absolute minimum value is $y(-\frac{1}{b}) = -\frac{1}{e} \approx -.368, b \ne 0$. ☺

1998: AB-3



t (seconds)	$v(t)$ (feet per second)
0	0
5	12
10	20
15	30
20	55
25	70
30	78
35	81
40	75
45	60
50	72

The graph of the velocity $v(t)$, in ft/sec, of a car traveling on a straight road, for $0 \le t \le 50$, is shown above. A table of values for $v(t)$, at 5 second intervals of time t, is shown to the right of the graph.

(a) During what intervals of time is the acceleration of the car positive? Give a reason for your answer.

(b) Find the average acceleration of the car, in ft/sec², over the interval $0 \le t \le 50$.

(c) Find one approximation for the acceleration of the car, in ft/sec², at $t = 40$. Show the computations you used to arrive at your answer.

(d) Approximate $\int_0^{50} v(t)\,dt$ with a Riemann sum, using the midpoints of five subintervals of equal length. Using correct units, explain the meaning of this integral.

(a) Positive acceleration implies increasing velocity. $a > 0$ for $\{t \mid 0 < t < 35 \text{ or } 45 < t < 50\}$. ☺

(b) $\bar{a} = \frac{v(50) - v(0)}{50 - 0} = \frac{72 - 0}{50 - 0} = \frac{36}{25} = 1.44 \text{ ft/sec}^2$ ☺

(c) $a(40)$ = slope of tangent to graph of $v(t)$ at t = 40.

$a(40) \approx \frac{v(45) - v(35)}{45 - 35} = \frac{60 - 81}{10} = -2.1 \text{ ft/sec}^2$ (symmetric difference quotient)

OR $a(40) \approx \frac{v(35) - v(40)}{35 - 40} = \frac{81 - 75}{-5} = -1.2 \text{ ft/sec}^2$ **OR** $a(40) \approx \frac{v(45) - v(40)}{45 - 40} = \frac{60 - 75}{5} = -3 \text{ ft/sec}^2$ ☺

(d) $\Delta t = 10 \text{ sec} \Rightarrow S = 10[v(5) + v(15) + v(25) + v(35) + v(45)] = 2530$ feet.

The Riemann sum represents the distance travelled from t = 0 to t = 50 in feet. ☺

1998: AB-4

Let f be a function with $f(1) = 4$ such that for all points (x, y) on the graph of f the slope is given by $\dfrac{3x^2 + 1}{2y}$.

(a) Find the slope of the graph of f at the point where $x = 1$.

(b) Write an equation for the line tangent to the graph of f at $x = 1$ and use it to approximate $f(1.2)$.

(c) Find $f(x)$ by solving the separable differential equation $\dfrac{dy}{dx} = \dfrac{3x^2 + 1}{2y}$ with the initial condition $f(1) = 4$.

(d) Use your solution from part (c) to find $f(1.2)$.

(a) $\dfrac{dy}{dx} = \dfrac{3x^2 + 1}{2y}$ and $f(1) = 4 \Longrightarrow \left.\dfrac{dy}{dx}\right|_{(1,4)} = \dfrac{1}{2}$ ☺

(b) The equation of the tangent: $y - 4 = \dfrac{1}{2}(x - 1)$ or $y = \dfrac{1}{2}x + \dfrac{7}{2}$ ☺

$f(1.2) \approx \dfrac{1}{2}(1.2) + \dfrac{7}{2} = 4.1$ ☺

(c) $\dfrac{dy}{dx} = \dfrac{3x^2 + 1}{2y} \Longrightarrow \int 2y\,dy = \int (3x^2 + 1)\,dx \Longrightarrow y^2 = x^3 + x + C$

$f(1) = 4 \Longrightarrow C = 14 \Longrightarrow f(x) = \sqrt{x^3 + x + 14}$ ☺

(d) $f(1.2) \approx 4.114$ ☺

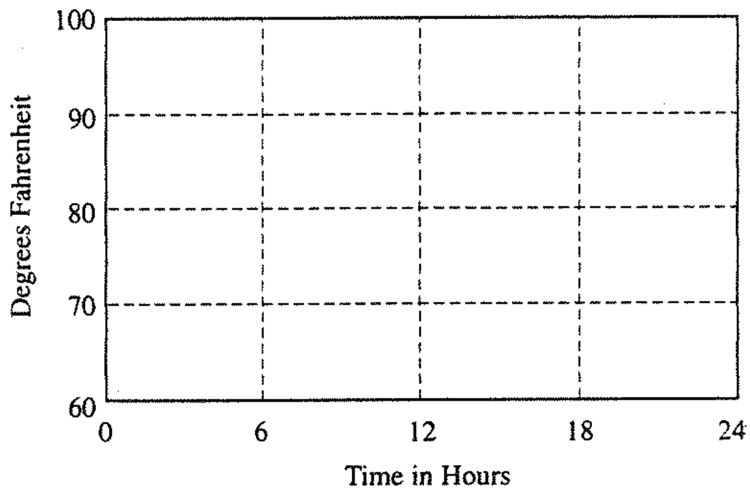
1998: AB-5; BC-5

The temperature outside a house during a 24-hour period is given by

$$F(t) = 80 - 10\cos\left(\frac{\pi t}{12}\right),\quad 0 \le t \le 24,$$

where $F(t)$ is measured in degrees Fahrenheit and t is measured in hours.

(a) Sketch the graph of F on the grid below.



(b) Find the average temperature, to the nearest degree Fahrenheit, between $t = 6$ and $t = 14$.

(c) An air conditioner cooled the house whenever the outside temperature was at or above 78 degrees Fahrenheit. For what values of t was the air conditioner cooling the house?

(d) The cost of cooling the house accumulates at the rate of \$0.05 per hour for each degree the outside temperature exceeds 78 degrees Fahrenheit. What was the total cost, to the nearest cent, to cool the house for this 24-hour period?

(b) $F_{avg}(t) = \frac{1}{14-6}\int_6^{14}\left(80 - 10\cos\left(\frac{\pi t}{12}\right)\right)dt$

$\approx 87.162° \approx 87°$ ☺

(c) Use calculator to solve:

$80 - 10\cos\left(\frac{\pi t}{12}\right) \ge 78$

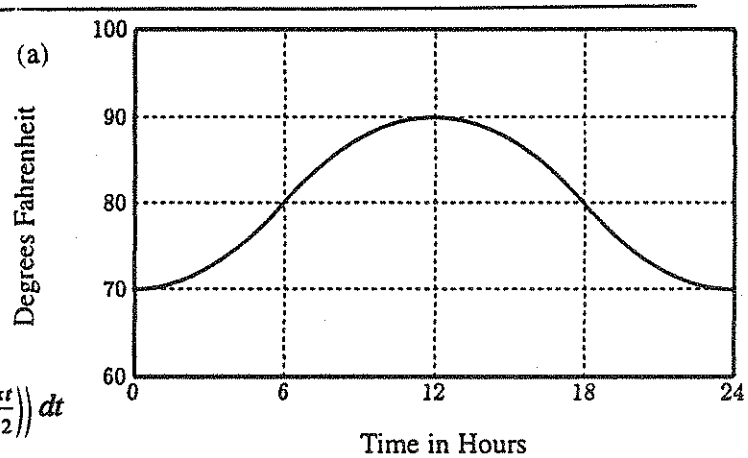
$5.231 \le t \le 18.769$ ☺

(d) $F_{avg}(t) = \frac{1}{18.769 - 5.231}\int_{5.231}^{18.769}\left(80 - 10\cos\left(\frac{\pi t}{12}\right)\right)dt$

$\approx 85.529°$ ★

Cost $\approx .05(18.769 - 5.231)(85.529 - 78) \approx \$5.10$ ☺

(a)



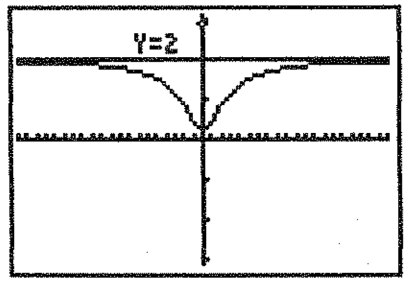
1998: AB-6

Consider the curve defined by $2y^3 + 6x^2y - 12x^2 + 6y = 1$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{4x - 2xy}{x^2 + y^2 + 1}$.

(b) Write an equation of each horizontal tangent line to the curve.

(c) The line through the origin with slope -1 is tangent to the curve at point P. Find the x- and y-coordinates of point P.



(a) $6y^2\dfrac{dy}{dx} + \left(6x^2\dfrac{dy}{dx} + 12xy\right) - 24x + 6\dfrac{dy}{dx} = 0$

$\left(6y^2 + 6x^2 + 6\right)\dfrac{dy}{dx} = 24x - 12xy$

$\dfrac{dy}{dx} = \dfrac{4x - 2xy}{y^2 + x^2 + 1}$ ☺

$2y^3 + 6x^2y - 12x^2 + 6y = 1$

$x = \pm\sqrt{\dfrac{1 - 6y - 2y^3}{6y - 12}}$ $\quad .1651648005 \le y < 2$

(b) Tangent line is horizontal *iff* $\dfrac{dy}{dx} = 0$ *iff* $\left(4x - 2xy = 0 \text{ and } y^2 + x^2 + 1 \ne 0\right)$

$4x - 2xy = 2x(2 - y) = 0 \Longrightarrow (x = 0 \text{ or } y = 2)$

$y = 2 \Rightarrow$ no value for x ($y = 2$ is not a tangent; it is an asymptote.)

$x = 0 \Longrightarrow 2y^3 + 6y - 1 = 0$ This equation has exactly one solution (by graphical solution or Descartes' Rule of Signs).

Using the calculator: $x = 0 \Longrightarrow y \approx .165 \Longrightarrow$ equation of tangent is $y = .165$ ☺

(c) Equation of the line: $y = -x$

At the point of tangency: $\dfrac{4x - 2xy}{x^2 + y^2 + 1} = -1$ and $y = -x$

$\Longrightarrow 4x^2 + 4x + 1 = 0 \Longrightarrow x = -\dfrac{1}{2} \Longrightarrow y = \dfrac{1}{2} \Longrightarrow P\left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$ ☺

Let R be the region in the first quadrant bounded by the graph of $y = 8 - x^{\frac{3}{2}}$, the x-axis, and the y-axis.

(a) Find the area of the region R.

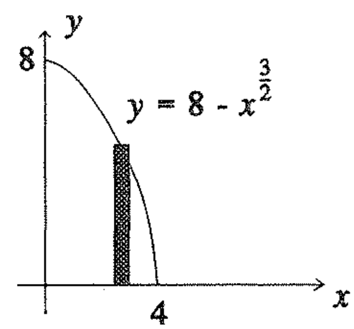
(b) Find the volume of the solid generated when R is revolved about the x-axis.

(c) The vertical line $x = k$ divides the region R into two regions such that when these two regions are revolved about the x-axis, they generate solids with equal volumes. Find the value of k.

(a) $A = \int_0^4 \left(8 - x^{\frac{3}{2}}\right) dx$

$= 8x - \frac{2}{5}x^{\frac{5}{2}}\Big|_0^4$

$= \left(\left(32 - \frac{64}{5}\right) - 0\right) = \frac{96}{5} = 19.2 \text{ units}^2$ ☺



(b) disks: $V = \pi \int_0^4 \left(8 - x^{\frac{3}{2}}\right)^2 dx = 115.2\pi \approx 361.911 \text{ units}^3$ ☺

(c) $\pi \int_0^k \left(8 - x^{\frac{3}{2}}\right)^2 dx = \frac{1}{2}(115.2\pi)$

$\int_0^k \left(64 - 16x^{\frac{3}{2}} + x^3\right) dx = 57.6$

$64x - \frac{32}{5}x^{\frac{5}{2}} + \frac{1}{4}x^4\Big|_0^k = 57.6$

$64k - \frac{32}{5}k^{\frac{5}{2}} + \frac{1}{4}k^4 = 57.6$

Using the calculator: $k \approx .995$ ☺

1998: BC-3

Let f be a function that has derivatives of all orders for all real numbers. Assume $f(0) = 5$, $f'(0) = -3$, $f''(0) = 1$, and $f'''(0) = 4$.

(a) Write the third-degree Taylor polynomial for f about $x = 0$ and use it to approximate $f(0.2)$.

(b) Write the fourth-degree Taylor polynomial for g, where $g(x) = f(x^2)$, about $x = 0$.

(c) Write the third-degree Taylor polynomial for h, where $h(x) = \int_0^x f(t)\,dt$, about $x = 0$.

(d) Let h be defined as in part (c). Given that $f(1) = 3$, either find the exact value of $h(1)$ or explain why it cannot be determined.

(a) $f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$

$f(x) \approx 5 - 3x + \frac{1}{2}x^2 + \frac{2}{3}x^3$ ☺

$f(0.2) \approx 4.425$ ☺

(b) $g(x) = f(x^2) \approx 5 - 3x^2 + \frac{1}{2}x^4$ ☺

(c) $h(x) = \int_0^x f(t)\,dt \approx \int_0^x \left(5 - 3t + \frac{1}{2}t^2\right) dt$

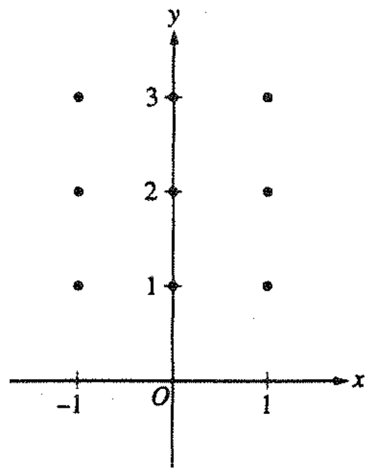
$\approx 5t - \frac{3}{2}t^2 + \frac{1}{6}t^3 \Big|_0^x \approx 5x - \frac{3}{2}x^2 + \frac{1}{6}x^3$ ☺

(d) The *exact* value of $h(1)$ cannot be determined because $f(t)$ is not defined for $0 < t < 1$; it is defined only for $t = 0$ and $t = 1$. ☺

1998: BC-4

Consider the differential equation given by $\frac{dy}{dx} = \frac{xy}{2}$.

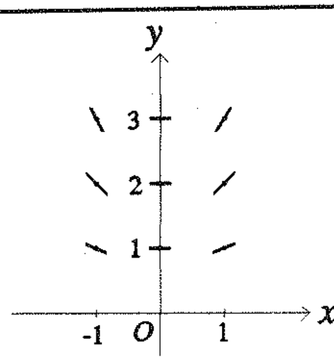
(a) On the axes provided below, sketch a slope field for the given differential equation at the nine points indicated.



(b) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(0) = 3$. Use Euler's method starting at $x = 0$, with a step size of 0.1, to approximate $f(0.2)$. Show the work that leads to your answer.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = 3$. Use your solution to find $f(0.2)$.

(a)



(b) Start at (0 , 3). $\frac{dy}{dx} = 0 \Longrightarrow \Delta y \approx \frac{dy}{dx}\Delta x = 0 \Longrightarrow f(.1) \approx 3 + 0 = 3.$

At (.1 , 3), $\frac{dy}{dx} = .15 \Longrightarrow \Delta y \approx (.15)(.1) = .015 \Longrightarrow f(.2) \approx 3 + .015 = 3.015$ ☺

(c) $\frac{dy}{dx} = \frac{xy}{2} \Longrightarrow \int \frac{1}{y}\,dy = \int \frac{x}{2}\,dx \Longrightarrow \ln|y| = \frac{1}{4}x^2 + C \Longrightarrow y = C_1 e^{\frac{x^2}{4}}$

$f(0) = 3 \Longrightarrow C_1 = 3 \Longrightarrow y = f(x) = 3e^{\frac{x^2}{4}}$ ☺

$f(0.2) = 3e^{.01} \approx 3.030$ ☺

1998: BC-6

A particle moves along the curve defined by the equation $y = x^3 - 3x$. The x-coordinate of the particle, $x(t)$, satisfies the equation $\frac{dx}{dt} = \frac{1}{\sqrt{2t+1}}$, for $t \geq 0$ with initial condition $x(0) = -4$.

(a) Find $x(t)$ in terms of t.

(b) Find $\frac{dy}{dt}$ in terms of t.

(c) Find the location and speed of the particle at time $t = 4$.

(a) $x(t) = \int \frac{1}{\sqrt{2t+1}}\,dt = \frac{1}{2}\int u^{-\frac{1}{2}}du = u^{\frac{1}{2}} = \sqrt{2t+1} + C$

$(u = 2t + 1,\ \frac{1}{2}du = dt)$

$x(0) = -4 \Longrightarrow C = -5 \Longrightarrow x(t) = \sqrt{2t+1} - 5$ ☺

(b) $y = x^3 - 3x \Longrightarrow \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = (3x^2 - 3)\left(\frac{1}{\sqrt{2t+1}}\right) = \frac{3(\sqrt{2t+1} - 5)^2 - 3}{\sqrt{2t+1}}$ ☺

(c) Location: $t = 4 \Longrightarrow x = -2$ and $y = -2 \Longrightarrow$ the particle is at $(-2, -2)$ ☺

Speed: $t = 4 \Longrightarrow$ speed $= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\left(\frac{1}{3}\right)^2 + 3^2} = \frac{\sqrt{82}}{3} \approx 3.018$ ☺

1999: AB-1

A particle moves along the y-axis with velocity given by $v(t) = t\sin(t^2)$ for $t \geq 0$.

(a) In which direction (up or down) is the particle moving at time $t = 1.5$? Why?

(b) Find the acceleration of the particle at time $t = 1.5$. Is the velocity of the particle increasing at $t = 1.5$? Why or why not?

(c) Given that $y(t)$ is the position of the particle at time t and that $y(0) = 3$, find $y(2)$.

(d) Find the total distance traveled by the particle from $t = 0$ to $t = 2$.

(a) $v(1.5) = 1.5\sin(1.5^2) \approx 1.167$

The particle is moving up at $t = 1.5$ because $v(1.5) > 0$. ☺

OR $v(t) > 0$ when $0 < t < \sqrt{\pi} \approx 1.772$.

Since $1.5 < \sqrt{\pi}$, $v(1.5) > 0 \Rightarrow$ the particle is moving up. ☺

(b) $v(t) = t\sin(t^2)$; $v'(t) = a(t)$

$a(t) = 2t^2\cos(t^2) + \sin(t^2) \Rightarrow a(1.5) \approx -2.049$ ☺

No. The velocity is decreasing at $t = 1.5$ because $a(1.5) < 0$. ☺

(c) $y(t) = \int v(t)dt = \int t\sin(t^2)dt = -\frac{1}{2}\cos(t^2) + C$

$y(0) = 3 \Rightarrow C = 3.5$

$y(t) = -\frac{1}{2}\cos(t^2) + 3.5 \Rightarrow y(2) \approx 3.827$ ☺

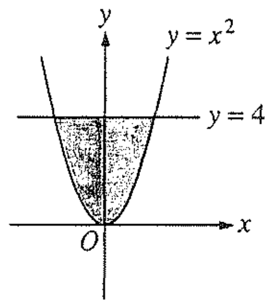
(d) Total distance $= \int_0^2 |v(t)|dt = \int_0^2 |t\sin(t^2)|dt \approx 1.173$ ☺

OR $v(t) = t\sin(t^2) = 0$ when $t = 0$ *or* $t = \sqrt{\pi} \approx 1.772$

$y(0) = 3$; $y(\sqrt{\pi}) = 4$; $y(2) = 3.827$

Total distance $= \left(y(\sqrt{\pi}) - y(0)\right) + \left(y(\sqrt{\pi}) - y(2)\right) \approx 1.173$ ☺

1999: AB-2; BC-2



The shaded region, R, is bounded by the graph of $y = x^2$ and the line $y = 4$, as shown in the figure above.

(a) Find the area of R.

(b) Find the volume of the solid generated by revolving R about the x-axis.

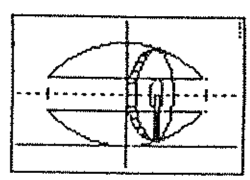
(c) There exists a number k, $k > 4$, such that when R is revolved about the line $y = k$, the resulting solid has the same volume as the solid in part (b). Write, but do not solve, an equation involving an integral expression that can be used to find the value of k.

(a) Points of intersection: $x^2 = 4 \implies x = \pm 2$

Area of $R = \int_{-2}^{2} (4 - x^2)\,dx \approx 10.667$ ☺

(b) Washers: $V = \pi \int_{-2}^{2} \left[(4)^2 - (x^2)^2\right] dx = 51.2\pi \approx 160.850$ ☺

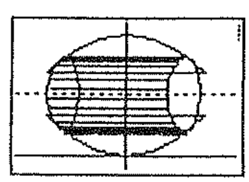
(c) Washers: $V_k = \pi \int_{-2}^{2} \left[(k - x^2)^2 - (k-4)^2\right] dx = 51.2\pi$



$y = k$

OR

Cylindrical shells: $V_k = 2\pi \int_{0}^{4} (k - y)(2\sqrt{y})\,dy = 51.2\pi$ ☺



$y = k$

1999: AB-3; BC-3

t (hours)	$R(t)$ (gallons per hour)
0	9.6
3	10.4
6	10.8
9	11.2
12	11.4
15	11.3
18	10.7
21	10.2
24	9.6

The rate at which water flows out of a pipe, in gallons per hour, is given by a differentiable function R of time t. The table above shows the rate as measured every 3 hours for a 24-hour period.

(a) Use a midpoint Riemann sum with 4 subdivisions of equal length to approximate $\int_0^{24} R(t)dt$. Using correct units, explain the meaning of your answer in terms of water flow.

(b) Is there some time t, $0 < t < 24$, such that $R'(t) = 0$? Justify your answer.

(c) The rate of water flow $R(t)$ can be approximated by $Q(t) = \frac{1}{79}\left(768 + 23t - t^2\right)$. Use $Q(t)$ to approximate the average rate of water flow during the 24-hour time period. Indicate units of measure.

(a) $\int_0^{24} R(t)\,dt \approx R(3)\Delta x + R(9)\Delta x + R(15)\Delta x + R(21)\Delta x$

$$\approx (10.4)6 + (11.2)6 + (11.3)6 + (10.2)6 = 258.6 \text{ gal.}$$ ☺

The Riemann sum estimates the total water flow (in gallons) during the 24 hour period. ☺

(b) Yes. There exists at least one time t, $0 < t < 24$, such that $R'(t) = 0$. Since $R(t)$ is differentiable, it is continuous. By Rolle's Theorem, if a function is continuous on a closed interval $[a,b]$, differentiable on the open interval (a,b), and $f(a) = f(b)$ [note: $R(0) = R(24)$], then there exists an $x \in (a,b)$ such that $f'(x) = 0$. ☺

(c) $R(t) \approx Q(t) = \frac{1}{79}(768 + 23t - t^2)$

$$R_{avg}(t) \approx Q_{avg}(t) = \frac{1}{24-0}\int_0^{24} \frac{1}{79}(768 + 23t - t^2)\,dt \approx 10.785 \text{ gal/hr}$$ ☺

1999: AB-4

Suppose that the function f has a continuous second derivative for all x, and that $f(0) = 2$, $f'(0) = -3$, and $f''(0) = 0$. Let g be a function whose derivative is given by $g'(x) = e^{-2x}(3f(x) + 2f'(x))$ for all x.

(a) Write an equation of the line tangent to the graph of f at the point where $x = 0$.

(b) Is there sufficient information to determine whether or not the graph of f has a point of inflection when $x = 0$? Explain your answer.

(c) Given that $g(0) = 4$, write an equation of the line tangent to the graph of g at the point where $x = 0$.

(d) Show that $g''(x) = e^{-2x}(-6f(x) - f'(x) + 2f''(x))$. Does g have a local maximum at $x = 0$? Justify your answer.

(a) $y - 2 = -3(x - 0)$ OR $y = -3x + 2$ ☺

(b) No. There is not sufficient information to determine whether or not the graph of f has a point of inflection where $x = 0$. While $f''(x) = 0$, it is not a *sufficient* condition since there is no way of verifying that $f''(x)$ changes sign about $x = 0$. ☺

(c) $g'(0) = e^0(3f(0) + 2f'(0)) = 0$

Therefore, the equation of the tangent at (0,4) is $y = 4$. ☺

(d) $g'(x) = e^{-2x}(3f(x) + 2f'(x))$

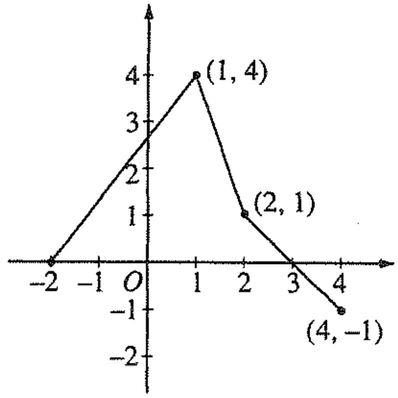
$$\begin{aligned} g''(x) &= e^{-2x}(3f'(x) + 2f''(x)) + (-2e^{-2x})(3f(x) + 2f'(x)) \\ &= e^{-2x}(3f'(x) + 2f''(x) - 6f(x) - 4f'(x)) \\ &= e^{-2x}(-6f(x) - f'(x) + 2f''(x)) \end{aligned}$$ ☺

$x = 0$ is a critical value ($g'(0) = 0$ — see part (c)).

$g''(0) = e^0(-6(2) - (-3) + 2(0)) = -9 < 0$

Therefore, by the second derivative test, g has a local (relative) maximum at $x = 0$.

1999: AB-5; BC-5



The graph of the function f, consisting of three line segments, is given above. Let $g(x) = \int_1^x f(t)dt$.

(a) Compute $g(4)$ and $g(-2)$.

(b) Find the instantaneous rate of change of g, with respect to x, at $x = 1$.

(c) Find the absolute minimum value of g on the closed interval $[-2, 4]$. Justify your answer.

(d) The second derivative of g is not defined at $x = 1$ and $x = 2$. How many of these values are x-coordinates of points of inflection of the graph of g? Justify your answer.

(a) $g(-2) = \int_1^{-2} f(t)\,dt = -\int_{-2}^{1} f(t)\,dt = -\frac{1}{2}(3)(4) = -6$ ☺

$g(4) = \int_1^4 f(t)\,dt = \int_1^2 f(t)\,dt + \int_2^3 f(t)\,dt + \int_3^4 f(t)\,dt$

$= \frac{1}{2}(1)(1+4) + \frac{1}{2}(1)(1) - \frac{1}{2}(1)(1) = 2.5$ ☺

(b) By the Second Fundamental Theorem of Calculus, $g'(x) = f(x) \implies g'(1) = f(1) = 4$ ☺

(c) $g'(x) = f(x) = 0 \Rightarrow x = 3$ is a critical value. Since g is increasing on $[-2,3]$ and decreasing on $[3,4]$, $g(3)$ is an absolute maximum. Therefore, the absolute minimum must occur at an endpoint extremum. Using the results of part (a), the minimum value of $g(x) = -6$. ☺

(d) One; $x = 1$.

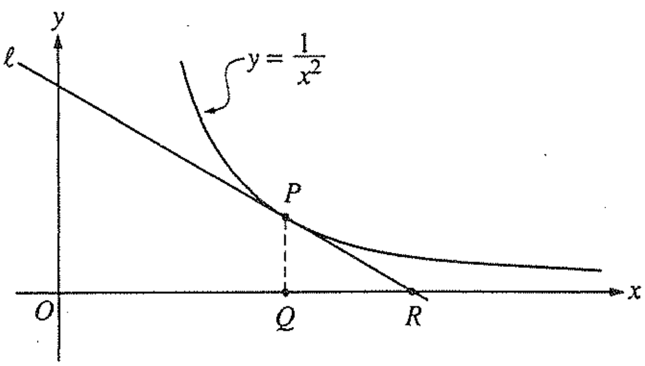
$g''(x) = f'(x) > 0$ on $(-2,1)$

$g''(x) = f'(x) < 0$ on $(1,2)$

$g''(x) = f'(x) < 0$ on $(2,4)$

Therefore, there is a point of inflection at just $(1, g(1))$. ☺

1999: AB-6



In the figure above, line ℓ is tangent to the graph of $y = \frac{1}{x^2}$ at point P, with coordinates $\left(w, \frac{1}{w^2}\right)$, where $w > 0$. Point Q has coordinates $(w, 0)$. Line ℓ crosses the x-axis at point R, with coordinates $(k, 0)$.

(a) Find the value of k when $w = 3$.

(b) For all $w > 0$, find k in terms of w.

(c) Suppose that w is increasing at the constant rate of 7 units per second. When $w = 5$, what is the rate of change of k with respect to time?

(d) Suppose that w is increasing at the constant rate of 7 units per second. When $w = 5$, what is the rate of change of the area of ΔPQR with respect to time? Determine whether the area is increasing or decreasing at this instant.

(a) $y = \frac{1}{x^2} \Rightarrow \frac{dy}{dx} = -\frac{2}{x^3}$; $\left.\frac{dy}{dx}\right|_{x=3} = -\frac{2}{27}$

Line l through $(k,0)$ and $\left(3, \frac{1}{9}\right)$ has slope $\frac{0-\frac{1}{9}}{k-3} = \frac{-2}{27} \Rightarrow k = \frac{9}{2}$. ☺

(b) Line l through $(k,0)$ *and* $\left(w, \frac{1}{w^2}\right)$ has slope $-\frac{2}{w^3}$

Then $\frac{0-\frac{1}{w^2}}{k-w} = \frac{-2}{w^3} \Rightarrow k = \frac{3w}{2}$ ☺

(c) $\frac{dk}{dt} = \frac{3}{2}\frac{dw}{dt} = \frac{3}{2}(7) = \frac{21}{2}$ units/sec ☺

(d) Area of $\Delta PQR = \frac{1}{2}(k-w)\frac{1}{w^2} = \frac{1}{2}\left(\frac{3}{2}w - w\right)\frac{1}{w^2} = \frac{1}{4}w^{-1}$

$\frac{dA}{dt} = -\frac{1}{4w^2}\frac{dw}{dt} = -\frac{1}{100}(7) = -\frac{7}{100}$ units^2/sec ☺

Since $\frac{dA}{dt} < 0$, the area of ΔPQR is decreasing. ☺

1999: BC-1

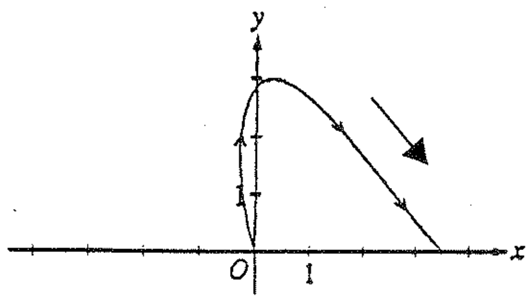
A particle moves in the xy-plane so that its position at any time t, $0 \le t \le \pi$, is given by $x(t) = \frac{t^2}{2} - \ln(1+t)$ and $y(t) = 3\sin t$.

(a) Sketch the path of the particle in the xy-plane below. Indicate the direction of motion along the path.

(b) At what time t, $0 \le t \le \pi$, does $x(t)$ attain its minimum value? What is the position $(x(t),\ y(t))$ of the particle at this time?

(c) At what time t, $0 < t < \pi$, is the particle on the y-axis? Find the speed and the acceleration vector of the particle at this time.

(a)



(b) $x'(t) = t - \dfrac{1}{1+t} = \dfrac{t^2+t-1}{1+t}$, $0 \le t \le \pi$

Critical values: $x'(t) = 0 \Rightarrow t = \dfrac{-1+\sqrt{5}}{2} \approx .618$

By the first derivative test (*see diagram*), $x(t)$ attains its minimum value at $t = \dfrac{-1+\sqrt{5}}{2} \approx .618$.

	0		$\frac{-1+\sqrt{5}}{2}$		π
$x'(t)$:		$-$		$+$	
$x(t)$:		decr.		incr.	

Since $x(.618) \approx -.290$ and $y(.618) \approx 1.738$, the particle's position at $t \approx .618$ is (-.290, 1.738).

(c) $x(t) = 0 \Rightarrow t \approx 1.2858$

$x'(t) = t - \frac{1}{1+t}$ and $y'(t) = 3\cos t$

All vector components are rounded to three decimal places.

speed $= |\vec{v}| = \sqrt{(x'(1.2858))^2 + (y'(1.2858))^2} = \sqrt{.848^2 + .843^2} \approx 1.196$

$x''(t) = 1 + \dfrac{1}{(1+t)^2}$ and $y''(t) = -3\sin t$

$\vec{a} = \langle x''(1.2858), y''(1.2858) \rangle = \langle 1.191, -2.879 \rangle$ ☺

The function f has derivatives of all orders for all real numbers x. Assume $f(2) = -3$, $f'(2) = 5$, $f''(2) = 3$, and $f'''(2) = -8$.

(a) Write the third-degree Taylor polynomial for f about $x = 2$ and use it to approximate $f(1.5)$.

(b) The fourth derivative of f satisfies the inequality $\left|f^{(4)}(x)\right| \le 3$ for all x in the closed interval $[1.5, 2]$. Use the Lagrange error bound on the approximation to $f(1.5)$ found in part (a) to explain why $f(1.5) \neq -5$.

(c) Write the fourth-degree Taylor polynomial, $P(x)$, for $g(x) = f\left(x^2 + 2\right)$ about $x = 0$. Use P to explain why g must have a relative minimum at $x = 0$.

(a) $$P_3(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3$$
$$= -3 + 5(x-2) + \frac{3}{2}(x-2)^2 - \frac{4}{3}(x-2)^3 \quad ☺$$
$f(1.5) \approx P_3(1.5) = -4.958\overline{3} \approx -4.958$ ☺

(b) $$\left|R_3(x,2)\right| = \left|f^4(c)\frac{(x-2)^4}{4!}\right| \le \left|\frac{3(x-2)^4}{24}\right|$$
$$\left|R_3(1.5,2)\right| \le \left|\frac{(-.5)^4}{8}\right| = .0078125 \approx .008$$

Therefore, the value of $f(1.5)$ is between $(-4.958 - .008)$ and $(-4.958 + .008)$.

Then $-4.966 \le f(1.5) \le -4.950 \Rightarrow f(1.5) \ne -5$ ☺

(c) $g(x) = f(x^2 + 2) \approx -3 + 5x^2 + \frac{3}{2}x^4 = P(x)$ ☺

The Taylor series for $g(x)$ about $x = 0$ is $g(x) = g(0) + g'(0)x + \frac{g''(0)}{2!}x^2 + \dots + \frac{g^{(n)}(0)}{n!}x^n + \dots$

The coefficient of x in $P(x) = g'(0) = 0 \Longrightarrow x = 0$ is a critical number of g.

The coefficient of x^2 in $P(x) = \frac{g''(0)}{2!} = 5 \Longrightarrow g''(0) = 10 > 0$.

Therefore, by the second derivative test, g has a relative minimum at $x = 0$. ☺

Let f be the function whose graph goes through the point (3, 6) and whose derivative is given by $f'(x) = \dfrac{1+e^x}{x^2}$.

(a) Write an equation of the line tangent to the graph of f at $x = 3$ and use it to approximate $f(3.1)$.

(b) Use Euler's method, starting at $x = 3$ with a step size of 0.05, to approximate $f(3.1)$. Use f'' to explain why this approximation is less than $f(3.1)$.

(c) Use $\int_3^{3.1} f'(x)dx$ to evaluate $f(3.1)$.

(a) $f'(x) = \dfrac{1+e^x}{x^2} \implies f'(3) = \dfrac{1+e^3}{9} \approx 2.343$

Equation of tangent: $y - 6 = 2.343(x-3)$ OR $y = 2.343x - 1.029$ ☺

$f(3.1) \approx y(3.1) = 6.234$ ☺

(b) $\Delta y = f'(x)\Delta x$, $\Delta x = .05$ and, initially, $x = 3$, $y = 6$

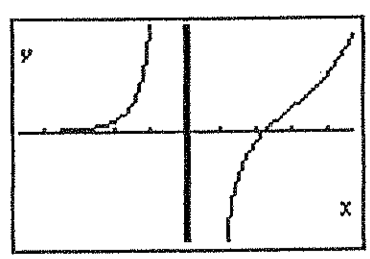
$\Delta y = \dfrac{1+e^3}{3^2}(.05) = .1171 \implies f(3.05) \approx 6 + .1171 = 6.1171$

$\Delta y = \dfrac{1+e^{3.05}}{3.05^2}(.05) = .1188 \implies f(3.1) \approx 6.1171 + .1188 = 6.2359 \approx 6.236$ ☺

$f''(x) = \dfrac{x^2e^x - 2x(1+e^x)}{x^4} = \dfrac{(x-2)e^x - 2}{x^3}$, $x \neq 0$

Since $f''(x) > 0$, $\forall x \geq 3$, f is locally concave up when $x \geq 3$. Therefore, the tangent segments used to approximate values of $f(x)$ are *below* the graph of f. This yields approximations less than the actual values of $f(x)$. The graph of f'' is shown below. ☺

$f''(x)$

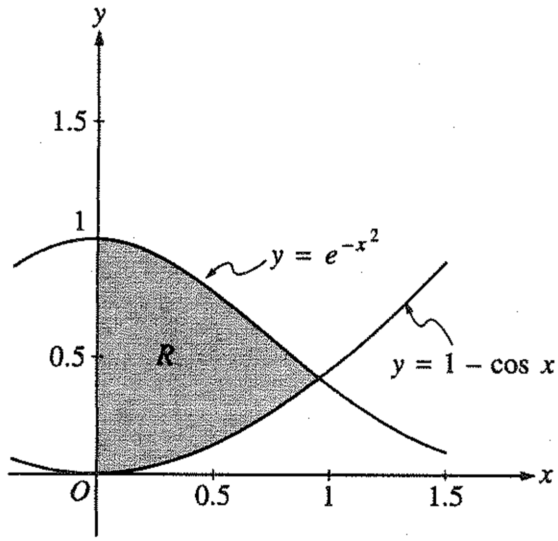


(c) $\int_3^{3.1} f'(x)\,dx = f(3.1) - f(3)$

$.2377 \approx f(3.1) - 6$

$f(3.1) \approx 6.2377 \approx 6.238$ ☺

2000: AB-1; BC-1



Let R be the shaded region in the first quadrant enclosed by the graphs of $y = e^{-x^2}$, $y = 1 - \cos x$, and the y-axis, as shown in the figure above.

(a) Find the area of the region R.

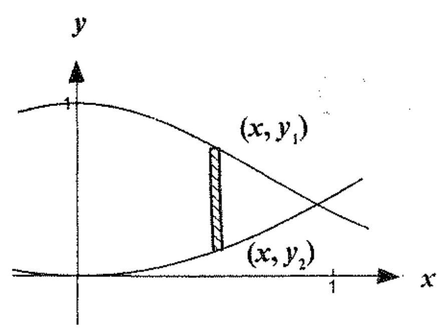
(b) Find the volume of the solid generated when the region R is revolved about the x-axis.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of this solid.

(a) $e^{-x^2} = 1 - \cos x$ at $x = 0.941944 = B$

$$A = \int_a^b \left[y_1(x) - y_2(x)\right] dx$$

$$A = \int_0^B \left[e^{-x^2} - (1 - \cos x)\right] dx \approx 0.591$$

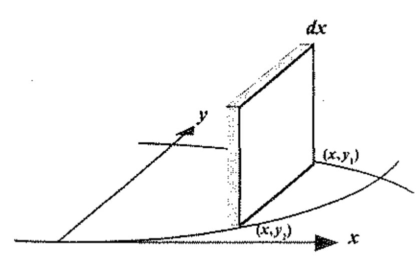


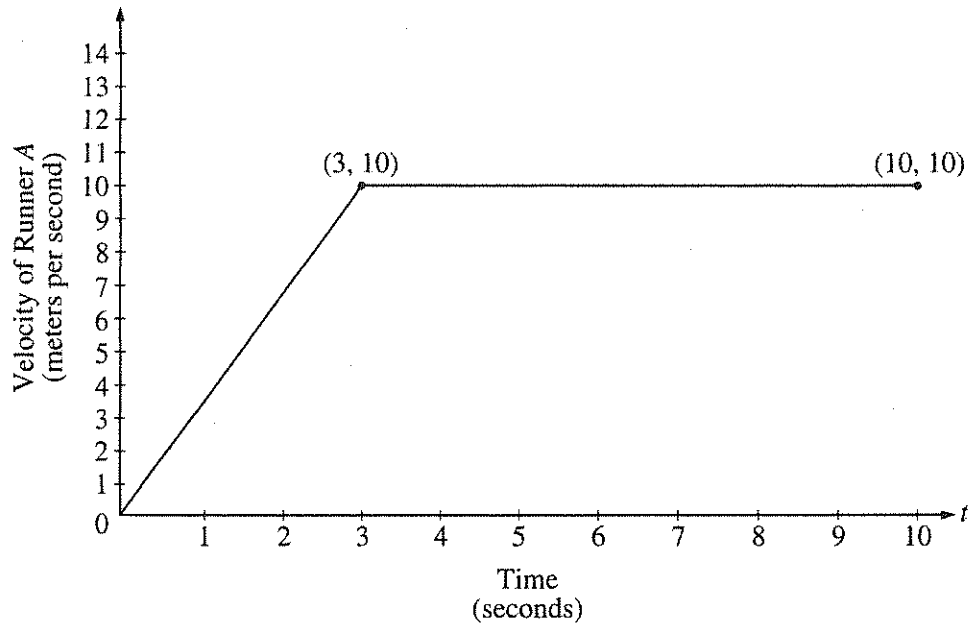
(b) $$\mathrm{V} = \pi \int_a^b \left[\left(y_1(x)\right)^2 - \left(y_2(x)\right)^2\right] dx$$

$$V = \pi \int_0^B \left[\left(e^{-x^2}\right)^2 - (1 - \cos x)^2\right] dx \approx 1.747$$

(c) $$V = \int_a^b \left[y_1(x) - y_2(x)\right]^2 dx$$

$$V = \int_0^B \left[e^{-x^2} - (1 - \cos x)\right]^2 dx \approx 0.461$$





Two runners, A and B, run on a straight racetrack for $0 \le t \le 10$ seconds. The graph above, which consists of two line segments, shows the velocity, in meters per second, of Runner A. The velocity, in meters per second, of Runner B is given by the function v defined by $v(t) = \dfrac{24t}{2t+3}$.

(a) Find the velocity of Runner A and the velocity of Runner B at time $t = 2$ seconds. Indicate units of measure.

(b) Find the acceleration of Runner A and the acceleration of Runner B at time $t = 2$ seconds. Indicate units of measure.

(c) Find the total distance run by Runner A and the total distance run by Runner B over the time interval $0 \le t \le 10$ seconds. Indicate units of measure.

(a) Since $v_A(t) = \begin{cases} \dfrac{10}{3}t\ , & 0 \le t \le 3 \\ 10\ , & 3 \le t \le 10 \end{cases}$, $v_A(2) = \dfrac{20}{3}$ m/s; $v_B(2) = \dfrac{48}{7}$ m/s ☺

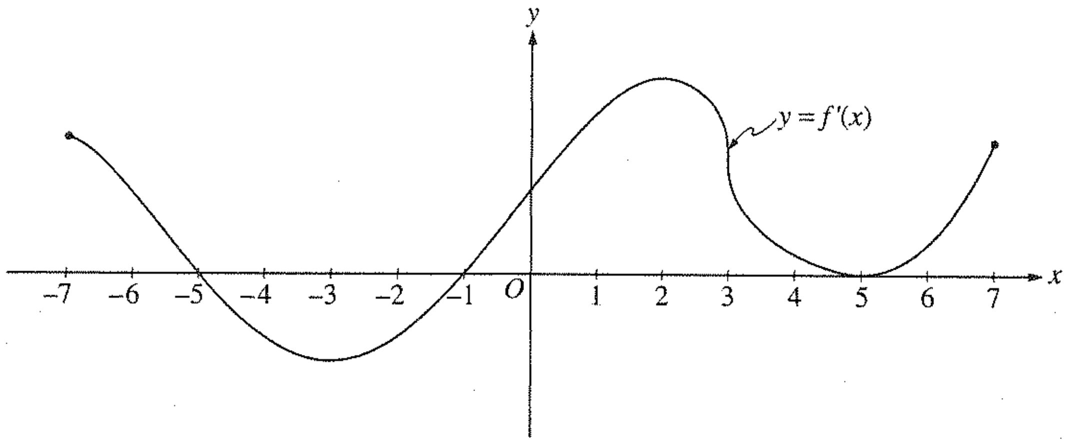
(b) From the graph: $a(t) = v'(t) \implies a_A(2) = \dfrac{10}{3}$ m/s^2 ☺

$$v_B(t) = \frac{24t}{2t+3} \implies a_B(t) = \frac{72}{(2t+3)^2} \implies a_B(2) = \frac{72}{49} \text{ m/s}^2$$ ☺

(c) $s_A = \int_0^{10} v_A(t)\,dt = \int_0^3 v_A(t)\,dt + \int_3^{10} v_A(t)\,dt = \frac{1}{2}(3)(10) + (7)(10) = 85$ m ☺

$$s_B = \int_0^{10} v_B(t)\,dt = \int_0^{10} \frac{24t}{2t+3}\,dt \approx 83.336 \text{ m}$$ ☺

2000: AB-3



The figure above shows the graph of f', the derivative of the function f, for $-7 \le x \le 7$. The graph of f' has horizontal tangent lines at $x = -3$, $x = 2$, and $x = 5$, and a vertical tangent line at $x = 3$.

(a) Find all values of x, for $-7 < x < 7$, at which f attains a relative minimum. Justify your answer.

(b) Find all values of x, for $-7 < x < 7$, at which f attains a relative maximum. Justify your answer.

(c) Find all values of x, for $-7 < x < 7$, at which $f''(x) < 0$.

(d) At what value of x, for $-7 \le x \le 7$, does f attain its absolute maximum? Justify your answer.

(a)

$f'(x)$		+		−		+		+		x
$f(x)$	-7	Incr	-5	Decr	-1	Incr	5	Incr	7	

There are critical numbers where $f' = 0$: $x = -5, -1, 5$. By the first derivative test, we look for critical numbers, c, for which $f' < 0$ when $x < c$ **and** $f' > 0$ when $x > c$. By the graph of f', this occurs at $x = -1$. Therefore, there is a relative minimum only at $x = -1$. ☺

(b) By the first derivative test, we look for critical numbers, c, for which $f' > 0$ when $x < c$ **and** $f' < 0$ when $x > c$. By the graph of f', this occurs at $x = -5$. Therefore, there is a relative maximum only at $x = -5$. ☺

(c) $f''(x) < 0 \implies f'(x)$ exists and is decreasing.
By the graph of f', $f''(x) < 0$ for $\{x \mid -7 < x < -3 \vee 2 < x < 5 \wedge x \neq 3\}$. ☺

(d) $x = 7$. The absolute maximum must occur at $x = -5$ or at an endpoint. $f(-5) > f(-7)$ because f is increasing on (-7,-5). The graph of f' shows that the magnitude of the negative change in f from $x = -5$ to $x = -1$ is smaller than the positive change in f from $x = -1$ to $x = 7$. Therefore, the net change in f is positive from $x = -5$ to $x = 7$, and $f(7) > f(-5)$. So, $f(7)$ is the absolute maximum. ☺

Water is pumped into an underground tank at a constant rate of 8 gallons per minute. Water leaks out of the tank at the rate of $\sqrt{t+1}$ gallons per minute, for $0 \le t \le 120$ minutes. At time $t = 0$, the tank contains 30 gallons of water.

(a) How many gallons of water leak out of the tank from time $t = 0$ to $t = 3$ minutes?

(b) How many gallons of water are in the tank at time $t = 3$ minutes?

(c) Write an expression for $A(t)$, the total number of gallons of water in the tank at time t.

(d) At what time t, for $0 \le t \le 120$, is the amount of water in the tank a maximum? Justify your answer.

Given $\frac{dA}{dt} = 8$ gal/min, $\frac{dL}{dt} = \sqrt{t+1}$ gal/min, and $A(0) = 30$.

(a) $L_3 = \int_0^3 (t+1)^{\frac{1}{2}}\,dt = \frac{2}{3}(t+1)^{\frac{3}{2}}\Big|_0^3 = \frac{14}{3}$

(b) $A(3) = 30 + (3)(8) - \frac{14}{3} = \frac{148}{3}$

(c) $A(t) = 30 + 8t - \int_0^t \sqrt{1+x}\,dx$ **OR**

$A(t) = \int A'(t)dt = \int\left[8 - (t+1)^{\frac{1}{2}}\right]dt = 8t - \frac{2}{3}(t+1)^{\frac{3}{2}} + C$

$A(0) = 30 \Rightarrow C = \frac{92}{3} \Rightarrow A(t) = 8t - \frac{2}{3}(t+1)^{\frac{3}{2}} + \frac{92}{3}$

(d) $A'(t) = 8 - \sqrt{1+t}$, $0 < t < 120$, implies that $A(t)$ is continuous on $[0,120]$. Critical values: $A'(t) = 0$ implies that $t = 63$. Since $A'(t) > 0$ when $0 < t < 63$ and $A'(t) < 0$ when $63 < t < 120$, $A(t)$ has an absolute maximum at $t = 63$. (First Derivative Test)

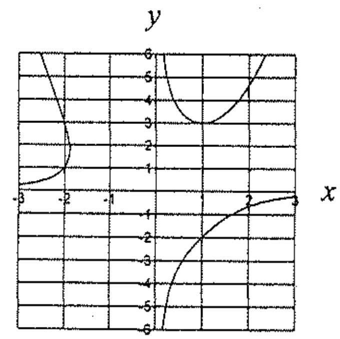
2000: AB-5; BC-5

Consider the curve given by $xy^2 - x^3y = 6$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{3x^2y - y^2}{2xy - x^3}$.

(b) Find all points on the curve whose x-coordinate is 1, and write an equation for the tangent line at each of these points.

(c) Find the x-coordinate of each point on the curve where the tangent line is vertical.



(a) $xy^2 - x^3y = 6 \implies x(2y)\dfrac{dy}{dx} + y^2 \cdot 1 - \left[x^3\dfrac{dy}{dx} + y(3x^2)\right] = 0$

$\implies \left(2xy - x^3\right)\dfrac{dy}{dx} = 3x^2y - y^2 \implies \dfrac{dy}{dx} = \dfrac{3x^2y - y^2}{2xy - x^3}$ ☺

(b) $x = 1 \implies y^2 - y = 6 \implies y = 3$ or $y = -2$

$\left.\dfrac{dy}{dx}\right|_{(1,3)} = 0 \implies$ the equation of the tangent is $y = 3$ ☺

$\left.\dfrac{dy}{dx}\right|_{(1,-2)} = 2 \implies$ the equation of the tangent is $y + 2 = 2(x - 1)$ ☺

(c) There is a vertical tangent where $\dfrac{dx}{dy} = 0$.

$\dfrac{dx}{dy} = \dfrac{2xy - x^3}{3x^2y - y^2} = 0$ *iff* $2xy - x^3 = 0$ and $3x^2y - y^2 \neq 0$

$\implies x\left(2y - x^2\right) = 0 \implies x = 0$ or $y = \dfrac{1}{2}x^2$. But $x \neq 0$ (See given equation).

$y = \dfrac{1}{2}x^2 \implies \dfrac{1}{4}x^5 - \dfrac{1}{2}x^5 = 6 \implies x^5 = -24 \implies x = \sqrt[5]{-24}$ ☺

Consider the differential equation $\dfrac{dy}{dx} = \dfrac{3x^2}{e^{2y}}$.

(a) Find a solution $y = f(x)$ to the differential equation satisfying $f(0) = \dfrac{1}{2}$.

(b) Find the domain and range of the function f found in part (a).

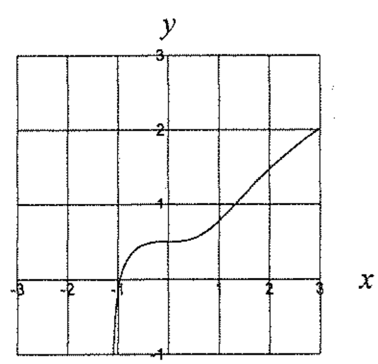
(a) $\dfrac{dy}{dx} = \dfrac{3x^2}{e^{2y}} \implies \displaystyle\int e^{2y}\,dy = \int 3x^2\,dx \implies \frac{1}{2}e^{2y} = x^3 + C$

$f(0) = \dfrac{1}{2} \implies C = \dfrac{1}{2}e \implies \dfrac{1}{2}e^{2y} = x^3 + \dfrac{1}{2}e$

$\implies e^{2y} = 2x^3 + e \implies 2y = \ln\left(2x^3 + e\right) \implies y = \dfrac{1}{2}\ln\left(2x^3 + e\right)$ ☺

(b) $2x^3 + e > 0 \implies x^3 > -\dfrac{e}{2} \implies$ the domain $= \left\{x \,\middle|\, x > \sqrt[3]{-\dfrac{e}{2}}\right\}$ ☺

Since $2x^3 + e$ takes on all positive real values, the range $= \mathbb{R}$ (the set of reals) ☺



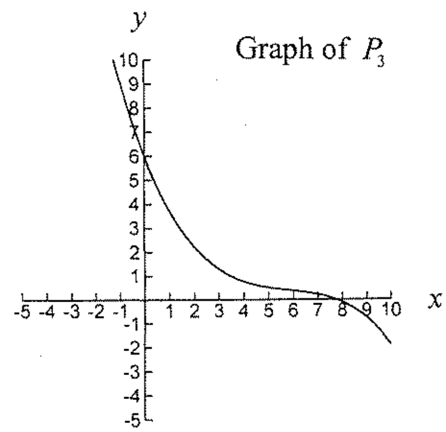
2000: BC-3

The Taylor series about $x = 5$ for a certain function f converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 5$ is given by $f^{(n)}(5) = \dfrac{(-1)^n n!}{2^n(n+2)}$, and $f(5) = \dfrac{1}{2}$.

(a) Write the third-degree Taylor polynomial for f about $x = 5$.

(b) Find the radius of convergence of the Taylor series for f about $x = 5$.

(c) Show that the sixth-degree Taylor polynomial for f about $x = 5$ approximates $f(6)$ with error less than $\dfrac{1}{1000}$.



(a) $P_3(x) = f(5) + f'(5)(x-5) + \dfrac{f''(5)}{2!}(x-5)^2 + \dfrac{f'''(5)}{3!}(x-5)^3$

$P_3(x) = \dfrac{1}{2} - \dfrac{1}{6}(x-5) + \dfrac{1}{16}(x-5)^2 - \dfrac{1}{40}(x-5)^3$ ☺

(b) Using the ratio test for absolute convergence:

$$\rho = \lim_{x\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{x\to\infty}\left|\frac{\dfrac{(-1)^{n+1}(n+1)!}{2^{n+1}(n+3)(n+1)!}(x-5)^{n+1}}{\dfrac{(-1)^n n!}{2^n(n+2)n!}(x-5)^n}\right|$$

$$= \lim_{x\to\infty}\left|\frac{(-1)^{n+1}(n+1)!}{2^{n+1}(n+3)(n+1)!}(x-5)^{n+1} \bullet \frac{2^n(n+2)n!}{(-1)^n n!(x-5)^n}\right| = \lim_{x\to\infty}\left|\frac{(-1)(n+2)}{2(n+3)}(x-5)\right| = \left|\frac{x-5}{2}\right|$$

$$\left|\frac{x-5}{2}\right| < 1 \implies -1 < \frac{x-5}{2} < 1 \implies -2 < x-5 < 2 \implies 3 < x < 7$$

Therefore, the radius of convergence = 2. ☺

(c) Since 6 is in the interval of convergence and this is a strictly alternating convergent series, the truncation error is less than $|a_7|$.

$$|a_7| = \left|\frac{(-1)^7 7!}{2^7(9)7!}(x-5)\right|_{x=6} = \left|\frac{1}{2^7(9)}\right| = \frac{1}{1152} < \frac{1}{1000}$$

Therefore, the error is less than $\dfrac{1}{1000}$ ☺

A moving particle has position $(x(t), y(t))$ at time t. The position of the particle at time $t = 1$ is $(2,6)$, and the velocity vector at any time $t > 0$ is given by $\left(1 - \frac{1}{t^2}, 2 + \frac{1}{t^2}\right)$.

(a) Find the acceleration vector at time $t = 3$.

(b) Find the position of the particle at time $t = 3$.

(c) For what time $t > 0$ does the line tangent to the path of the particle at $(x(t), y(t))$ have a slope of 8 ?

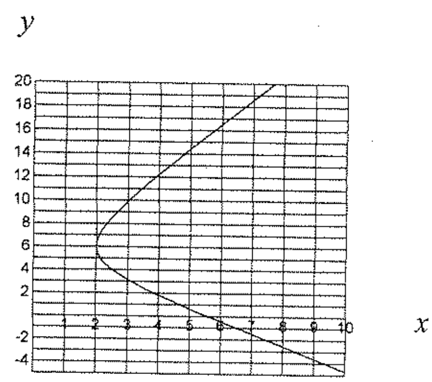
(d) The particle approaches a line as $t \to \infty$. Find the slope of this line. Show the work that leads to your conclusion.

(a) $\vec{a}(t) = \left\langle \frac{d(1 - t^{-2})}{dt}, \frac{d(2 + t^{-2})}{dt} \right\rangle = \langle 2t^{-3}, -2t^{-3} \rangle$

At $t = 3$, $\vec{a} = \left\langle \frac{2}{27}, -\frac{2}{27} \right\rangle$ ☺

(b) $\vec{r}(t) = \left\langle \int (1 - t^{-2})\,dt, \int (2 + t^{-2})\,dt \right\rangle = \langle t + t^{-1} + C_1, 2t - t^{-1} + C_2 \rangle$

$t = 1$, $\vec{r} = \langle 2, 6 \rangle \implies C_1 = 0$ and $C_2 = 5$



$\vec{r}(t) = \langle t + t^{-1}, 2t - t^{-1} + 5 \rangle$

at $t = 3$, $\vec{r} = \left\langle \frac{10}{3}, \frac{32}{3} \right\rangle$ ☺

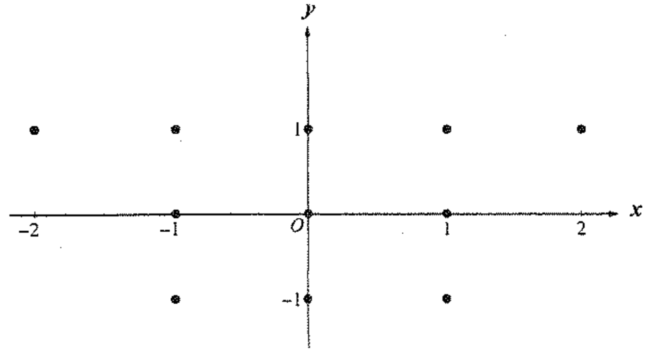
(c) $m = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2 + t^{-2}}{1 - t^{-2}} = 8 \implies t = \sqrt{\frac{3}{2}}$ ☺

(d) The slope of the asymptote $= \lim_{t \to \infty} \frac{dy/dt}{dx/dt} = \lim_{t \to \infty} \frac{2 + t^{-2}}{1 - t^{-2}} = 2$ ☺

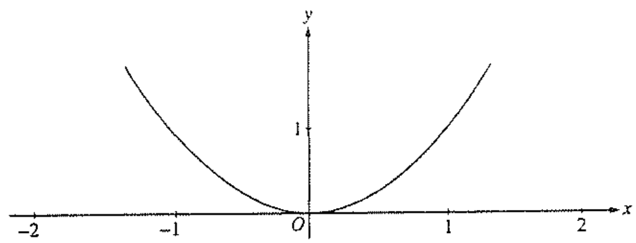
2000: BC-6

Consider the differential equation given by $\frac{dy}{dx} = x(y-1)^2$.

(a) On the axes provided, sketch a slope field for the given differential equation at the eleven points indicated.



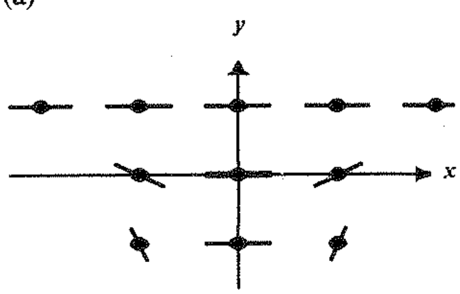
(b) Use the slope field for the given differential equation to explain why a solution could not have the graph shown below.

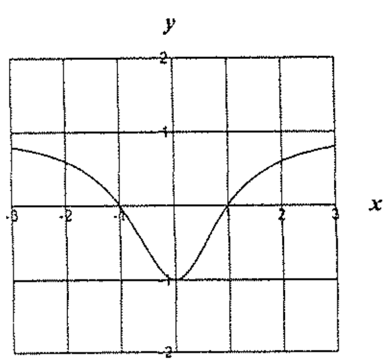


(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = -1$.

(d) Find the range of the solution found in part (c).

(a)





(b) Because when $y=1$, $\frac{dy}{dx}=0$ and in the given graph there is no horizontal tangent when $y=1$.

(c) $\frac{dy}{dx} = x(y-1)^2 \Rightarrow \int \frac{1}{(y-1)^2}\,dy = \int x\,dx \Rightarrow$

$$\frac{-1}{(y-1)} = \frac{1}{2}x^2 + C \Rightarrow y = 1 - \frac{1}{\frac{1}{2}x^2 + C}$$

$$f(0) = -1 \Rightarrow C = \tfrac{1}{2} \Rightarrow y = 1 - \frac{2}{x^2+1}$$

(d) The range of $y = f(x)$ is $\{y \mid -1 \le y < 1\}$

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